Walsh-Hadamard and Discrete Cosine Transform and Discrete Cosine Transform

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ABSTRACT: *A comparison of the developed complex Hadamard transform with the well-known unitary (orthogonal) transforms of Karhunen-Loeve, Fourier, Walsh-Hadamard and discrete cosine transform is presented. The comparison is made on the base of minimization of mean-squared error of reconstructed transform coefficients for two test images.*

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The use of unitary (orthogonal) transformations is inextricably linked with the development of modern digital image processing methods. Discrete unitary transforms described in [1], [2] and [3], have found applications in many areas of *N*-dimensional signal processing, spectral analysis, pattern recognition, digital coding, computational mathematics and etc. Stated simply, these transform coefficients that are small may be excluded from processing operations, such as filtering, compression and etc., without much loss in processing accuracy.

In this article a comparison of the developed complex Hadamard transform [4] with most used unitary transforms of Karhunen-Loeve (KLT), Fourier (DFT), Walsh-Hadamard (WHT) and discrete cosine transform (DCT) is presented.

There are various studies and comparisons of the orthogonal transforms [5], [6], [7], in which they discussed their properties, advantages and disadvantages from a statistical point of view. In this article a comparison of transformations in terms of the developed method [8] for optimal performance of the coefficients at the block coding for two test images was made.

The comparison is made through simulation of the developed algorithms for five unitary transforms – FFT, DCT,WHT, KLT and CHT on Matlab environment for two test images “Lena” and “Fruits” and the results are given in the experimental part.

2. Mathematical Description

The forward and the inverse 2D discrete unitary transform of sub-image $g(x,y)$ of size NxN can be expressed as the following equations:

$$S(u,v)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} g(x,y).r(x,y,u,v)$$
$$g(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} S(u,v).t(x,y,u,v) \quad (1)$$

In this equations $g(x,y)$ is the input image with spatial variables (x,y), $S(u,v)$ is forward transform with transform variables (u,v), $r(x,y,u,v)$ and $t(x,y,u,v)$ are called the forward and inverse transformation kernels, respectively. Because theinverse kernel $t(x,y,u,v)$ in (1) depends only on the indices (x,y,u,v) and not on the values of $g(x,y)$ and $S(u,v)$, it can be viewed as defining a set of basis functions or basis images [3].

This interpretation becomes clearer if the equation is modified in matrix form:

$$\mathbf{G}_{xy}=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} S(u,v)\mathbf{T}_{uv} \ , \quad (2)$$

where: $\mathbf{G}_{xy}$ is NxN matrix containing the pixels of $g(x, y)$, the matrices $\mathbf{T}_{uv}$ are the basis images and $S(u,v)$ are the spectral coefficients.

The forward transformation kernel is said to be separable if:

$$r(x, y, u, v) = r_1(x, u).r_2(y, v), \quad (3)$$

and the kernel is said to be symmetric if $r_1(x, u)$ functionally equal to $r_2(y, v)$, so that:

$$r(x, y, u, v) = r_1(x,u).r_1(y,v) \quad (4)$$

Identical comments apply to the inverse kernel by replacing function $r(x,y,u,v)$ with $t(x,y,u,v)$ in the equations (1).

As a sample the 2D Fourier transform has the following forward and inverse kernels:

$$r(x,y,u,v)=e^{-j2\pi(ux/M+vy/N)}$$
$$t(x,y,u,v)=\frac{1}{M.N}e^{j2\pi(ux/M+vy/N)} \ , \quad (5)$$

where $j=\sqrt{-1}$, so these kernels are complex.

A computationally simpler Walsh-Hadamard transform is derived from the following functionally identical kernels [3]:

$$r(x,y,u,v)=t(x,y,u,v)=\frac{1}{n}(-1)^{\sum_{i=0}^{m-1}[b_i(x)p_i(u)+b_i(y)p_i(v)]} \ , \quad (6)$$

where $n=2^m$. The summation in the exponent is performed in modulo 2 arithmetic and $b_z(k)$ is the k^{th} bit in the binary representation of z. The $p_i(l)$ is the conversion of $b_z(k)$ with code of Grey.

The most used discrete cosine transformation is obtained by following equal kernels:

$$r(x,y,u,v) = t(x,y,u,v) = \\ = a(u).a(v).cos\frac{(2x+1)u\pi}{2n} cos\frac{(2y+1)v\pi}{2n}, \tag{7}$$

where:

$$a(u) = \begin{cases} \sqrt{\frac{1}{n}} & \text{for } u = 0, \\ \sqrt{\frac{2}{n}} & \text{for } u = 1,2,.....n-1, \end{cases}$$

and similarly for *a(v)*.

The identical description for the developed complex Hadamard transformation kernels are made in [9] and are used for comparison.

The kernels of optimal Karhunen-Loeve transformation are calculated for each test image by calculation the correlation function and the preparation of the eigenvalues and eigenvectors as is given in [1]. To simplify the calculationsthe input learning vectors are taken from the current test image consecutively according to image linear scanning.

From the equations (1) to (4) the compared transformations can be generalized for two-dimensional signals (images) in the following way:

$$\left| \begin{array}{l} [Y] = [T_N][X][T_N] \\ [X] = \frac{1}{N^2}[T_N][Y][T_N] \end{array} \right. \tag{8}$$

where: $[X]$ is matrix of the input image with size NxN, $[T]$ is a matrix form of each transform kernel and the result is a spatial spectrum matrix $[Y]$ with the same size.

The symmetry of $[T]$ matrix coefficients allows 2D transforms to be accomplished in two steps. The first one is one-dimensional transform for every row of the image and the second one is one-dimensional transform for the columns. This difference of transformation makes easier the calculations and the symmetry guarantees that the correlations between image elements in horizontal and vertical direction will influence in the same way the determination of transformed elements. The same considerations can be made for two steps calculation of the inverse 2D transforms.

3. Experimental Results

The comparison is made on Matlab environment for five 2D unitary transforms – Karhunen-Loeve, Discrete Fourier Transform, Discrete Cosine Transform, Discrete Walsh- Hadamard Transform and Complex Hadamard Transform. The obtained experimental results for the test grayscale images “Lenna” and “Fruits” shown on the Fig.1b and Fig.1a with size 512x512 pixels and 8 bits per pixel. The simulations ware made for each image by transformation with kernel with size 8x8. The transform coefficients are reduced by the using of presented in [8] block truncation coding algorithm.

The obtained experimental results for the test images “Lenna” and “Fruits” (512x512, 8 bits) with sub-image kernel 8x8 are given on Table 1 and Table 2 respectively.

In the first part of each table the results for 48 reduced coefficients (the upper-left block with size 4x4 is saved), approximated with zero and mean values are shown, and in the second part the same experiments for 55 reduced coefficients (the upper-left block with size 3x3 - saved) are shown. The calculated values for the mean-square error (MSE), normalized mean-square error (NMSE), signal to noise ratio (SNR) and peak signal to noise ratio (PSNR) are shown.

Figure 1b. Test image “FRUITS” (512x512 pixels and 256 grey levels). The output images after inverse transform for 48 reduced coefficients are showed on Figure 2a and 2b respectively.

Reduction type	MSE	NMSE	SNR, dB	PSNR, dB
48 Reduced Coefficients, 16 saved				
Zero DFT	114.488	1.47 x10^{-5}	48.3180	27.5772
Mean DFT	103.124	1.32 x10^{-5}	48.7720	28.0312
Zero DCT	-0.3842	-4.9 x10^{-8}	73.0599	52.3191
Mean DCT	-0.2903	-3.7 x10^{-8}	74.2765	53.5357
Zero WHT	19.0409	2.45 x10^{-6}	56.1087	35.3679
Mean WHT	16.1269	2.07 x10^{-6}	56.8301	36.0893
Zero CHT	16.2735	2.09 x10^{-6}	56.7908	36.0500
Mean CHT	15.0354	1.93 x10^{-6}	57.1345	36.3936
Zero KLT	299.676	3.86 x10^{-5}	44.1391	23.3983
Mean KLT	249.875	3.21 x10^{-5}	44.9283	24.1875
55 Reduced Coefficients, 9 saved				
Zero DFT	126.300	1.62 x10^{-5}	47.8915	27.1507
Mean DFT	113.403	1.45 x10^{-5}	48.3593	27.6185
Zero DCT	-0.2014	-2.6 x10^{-8}	75.8649	55.1241
Mean DCT	-0.0054	-7.0 x10^{-10}	91.5437	70.8029
Zero WHT	48.4430	6.23 x10^{-6}	52.0533	31.3125
Mean WHT	45.8332	5.89 x10^{-6}	52.2938	31.5530
Zero CHT	46.2743	5.95 x10^{-6}	52.2522	31.5114
Mean CHT	42.5687	5.48 x10^{-6}	52.6147	31.8739
Zero KLT	313.557	4.03 x10^{-5}	43.9424	23.2016
Mean KLT	260.008	3.34 x10^{-5}	44.7557	24.0149

Table 1

Reduction type	MSE	NMSE	SNR, dB	PSNR, dB
48 Reduced Coefficients, 16 saved				
Zero DFT	20.2979	3.53 x10^{-6}	54.5211	35.0903
Mean DFT	17.7615	3.08 x10^{-6}	55.1008	35.6699
Zero DCT	-2.0464	-3.5 x10^{-7}	64.4855	45.0547
Mean DCT	-2.0174	-3.5 x10^{-7}	64.5477	45.1168
Zero WHT	3.7753	6.56 x10^{-7}	61.8261	42.3952
Mean WHT	0.8382	1.45 x10^{-7}	68.3620	48.9312
Zero CHT	0.8032	1.39 x10^{-7}	68.5470	49.1162
Mean CHT	0.3326	5.78 x10^{-8}	72.3756	52.9448
Zero KLT	61.9106	1.07 x10^{-5}	49.6779	30.2472
Mean KLT	55.0704	9.58 x10^{-6}	50.18.64	30.7556
55 Reduced Coefficients, 9 saved				
Zero DFT	22.203	3.86 x10^{-6}	54.1313	34.7005
Mean DFT	20.814	3.62 x10^{-6}	54.4119	34.9811
Zero DCT	-3.8473	-6.7 x10^{-7}	61.7440	42.3132
Mean DCT	-4.0750	-7.1 x10^{-7}	61.4943	42.0635
Zero WHT	5.2066	9.06 x10^{-7}	60.4300	40.9992
Mean WHT	3.7104	6.45 x10^{-7}	61.9014	42.4705
Zero CHT	7.1197	1.24 x10^{-6}	59.0710	39.6402
Mean CHT	6.1880	1.07 x10^{-6}	59.6801	40.2493
Zero KLT	62.011	1.07 x10^{-5}	49.6709	30.2400
Mean KLT	54.837	9.53 x10^{-6}	50.2048	30.7740

Table 2

Figure 1a. Test image “LENNA” (512x512 pixels and 256 gray levels)

Figure 1b. Test image “FRUITS” (512x512 pixels and 256 gray levels)

4. Conclusion

The general principles of comparison of 2D unitary transforms by the using of block transform coding of their coefficients of high order are given. The basic properties of CHT are discussed in previous publications. The obtained simulation results are practically identical for the CHT and real HT and show that both can be used in similar applications.

The best results are obtained for optimal Karhunen-Loeve transform and for the most used in compression standards discrete cosine transform. The results for the CHT are better than the DFT for small size of kernels. The main advantages of the developed algorithm for CHT are:

- Faster calculation compared to other transformations;

- Similar results with integer valued HT;

- Using the CHT instead most complicated Fourier transform and keep the possibilities for working with complex spectrum.

The developed Complex Hadamard Transform can be used in digital signal processing for spectral analysis, pattern recognition, digital watermarking, coding and transmission of one-dimensional and two-dimensional signals.

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