A Modification of Accumulate-Repeat-Accumulate Code Structure via Spatial Coupling

Shady M. Ibraheem[1], M. M. Abd Elrazzak[1], Salwa M. Serag Eldinm[2], W. Saad[3], Atef E. Aboelazm[3]
[1]Dep. of Communication Engineering
Faculty of Engineering, Mansoura University
Mansoura, Egypt
[2]Dep. of Communication Engineering
Faculty of Engineering, Taif University
Taif, KSA
[3]Dep. of Communication Engineering
Faculty of Electronic Engineering, Menoufiya University
Menouf, Egypt
{Shadymam, sserag}@gmail.com, mabdelrazzak@mans.edu.eg, waleedsaad100@yahoo.com, elazm40@hotmail.com

ABSTRACT: *In this paper a new family of accumulate repeat accumulate codes are established by a collection of interconnected proto-graphs in a spatially coupled manner. A little modification in the repeater-combiner stage of an accumulate repeat accumulate code emphasizes this development. Spatially coupled low density parity check (SC-LDPC) codes appear to approach the capacity universally across the binary-input memoryless (BMS) channels. However, the maximum degree distribution is unbounded and this leads to computational complexity problems at encoders and decoders. Accumulate repeat accumulate (ARA) codes could introduce bounded complexity ensembles that asymptotically achieve capacity on the binary erasure channels (BEC).*

So, we provide a density evolution (DE) analysis for systematic SC-ARA proto-graphs over the binary erasure channels (BEC). We also discuss the stability conditions for them. Simulation results show that over the BEC spatially coupling of ensembles of ARA codes drives the message-passing belief propagation decoding threshold (BP) to be closed to the maximum a posterior (MAP) threshold of the underlying codes.

Keywords: Binary Erasure Channels (BEC), Density Evolution (DE), Spatially Coupling, Degree Distributions, State Nodes

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1. Introduction

The capacity achieving error correcting codes attract much attention by the past ten years of research as they represent an optimal utilization of the channel coding reliability. Luby et al. [1] and Shokrollahi [2] introduced capacity achieving low density parity check (LDPC) codes whose complexities are linear in their block lengths on BEC.

Later, Jin et al. [3] initiated irregular repeat-accumulate (IRA) codes with lower encoding and decoding complexities over BEC. These codes still have *unbounded complexity* (per information bit) as the gap to capacity vanishes.

In [4], [5], Khandekar and McEliece discussed the decoding complexity of capacity approaching ensembles of irregular LDPC and IRA codes for the BEC. In [6], the authors conjecture capacity achieving ARA codes on BEC with *bounded density and complexity* per information bit. This result is achieved by puncturing bits and thereby retaining state nodes to represent the code.

The concept of spatially coupling was introduced in [7] for convergence-threshold improvement. The detailed convergence analysis of spatially coupled LDPC codes over BEC has been carried out in [8]. Further investigation and generalization can be shown in [9-10], [11]. Recently, those ideas are analytically investigated by Kudekar *et al.* In [12] where the authors couple together copies of a standard individual LDPC ensemble to construct a new chain-like ensemble. The chain has been terminated efficiently in [13].

In this paper we provide closed form DE equations for systematic SC-ARA codes from their proto-graphs. In general, ARA codes exhibit an outstanding performance over BEC at moderate block lengths. The spatially coupling approach is extended to these codes and threshold results are derived using DE equations. Then, we demonstrate the superiority of the performance for this construction emphasizing that theencoding and decoding density and complexity per information bit remains bounded as the gap to capacity vanishes.

The structure of the paper is as follows: Section II provides a preliminary on proto-graph ARA codes and their DE analysis for the BEC. Section III introduces the SC-ARA codes and their DE analysis via message passing algorithm. Section IV presents an explicit construction of capacity achieving SC-ARA codes with bounded density and complexity. Computer simulations exemplify our results in section V. Then, in section VI we conclude the paper.

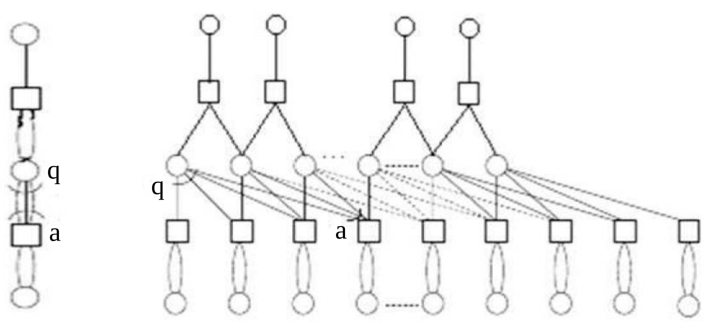


Figure 1. A proto-graph model of an ARA code and the chain connection

2. Preliminaries

2.1 Construction of ARA Codes

ARA codes can be considered as interleaved serially concatenated codes. An example of a proto-graph model of an ARA code is shown in figure 1. In this model there is one message bit node, a punctured bit node, at the top, a check node bit in the middle, a parity bit node at the bottom. A coupled chain of $2L + 1$ proto-graphs can be formed from the main proto-graph by connecting each punctured bit to l protograghs to the left and another l proto-graghs to the right. It is essential to add extra $2l$ parity check nodes to avoid degree-1 check nodes.

ARA codes have a low complexity encoding process. This process is achieved by the serial concatenation of an accumulator, a repetition code, an interleaver, a combiner and another accumulator.

For the coupled structure, the parity bit node at the *i*th location can be made to connect only message bits in the *i*th and previous locations.

2.2 Density Evolution of Systematic ARA Codes and Stability Conditions

As previously considered by Henry D. Pfister et al [6], the density evolution of a systematic ARA code on BEC with the fixed point analysis under iterative message-passing decoder could be obtained as follows:

Let L_i, R_i, λ_i, ρ_i be the fraction of "*punctured bit*" nodes with degree-*i*, the fraction of "*parity check B*" nodes with degree-*i*, the fraction of edges connected to degree-*i* "*punctured bit*" nodes and the fraction of edges connected to degree-*i* "*parity check B*" nodes, respectively. This is for the edges connecting the "*punctured bit*" nodes to the "*parity check B*" nodes. Their degree distributions will be $L(x) = \sum_{i=1}^{\infty} L_i x^i$, $R(x) = \sum_{i=1}^{\infty} R_i x^i$, $\lambda(x) = \sum_{i=1}^{\infty} \lambda_i x^{i-1}$, $\rho(x) = \sum_{i=1}^{\infty} \rho_i x^{i-1}$, respectively.

It can be proved that the relations

$$\lambda(x) = \frac{L'(x)}{L'(1)}, \rho(x) = \frac{R'(x)}{R'(1)} \tag{1}$$

or equivalently,

$$L(x) = \frac{\int_0^X \lambda(t)\, dt}{\int_0^1 \lambda(t)\, dt}, R(x) = \frac{\int_0^X \rho(t)\, dt}{\int_0^1 \rho(t)\, dt} \tag{2}$$

hold.

The design rate can be expressed in terms of degree distributions as

$$R = \frac{1}{1 + \frac{L'(1)}{R'(1)}} \tag{3}$$

Hence, from the Tanner graph of ARA codes by the assumption that the fraction of bits involved in finite-length cycles vanishes as the block length tends to infinity, the fixed point density evolution satisfies

$$x_i = \frac{\rho^2 \lambda \left(1 - \left(\frac{1-p}{1-pR(1-x_i)}\right)^2 \rho(1-x_i)\right)}{\left[1-(1-p)L\left(1-\left(\frac{1-p}{1-pR(1-x_i)}\right)\rho(1-x_i)\right)\right]^2} \tag{4}$$

Where p is the erasure probability of the transmitted codeword and x_i is the fixed point erasure probability at position *i*.

The recursion on (4) quickly results in very high order polynomials as the number of iteration is increased. However, to understand its behavior for small fixed point values of x_is, it may be effective to use the stability and instability conditions by taking the derivatives of RHS for $x_i = 0, 1$. As investigated in [6], this gives

$$\rho^2 \lambda_2 \left(\rho'(1) + \frac{2pR'(1)}{1-p}\right) < 1 \tag{5}$$

$$(1-p)^2 \rho^2 \left(\lambda'(1) + \frac{2(1-p)L'(1)}{p}\right) > 1 \tag{6}$$

for $x_i = 0$ to be stable and $x_i = 1$ to be unstable, respectively.

Note that ρ_2, λ_2 are the second coefficients of the distributions $\rho(x)$, $\lambda(x)$ respectively.

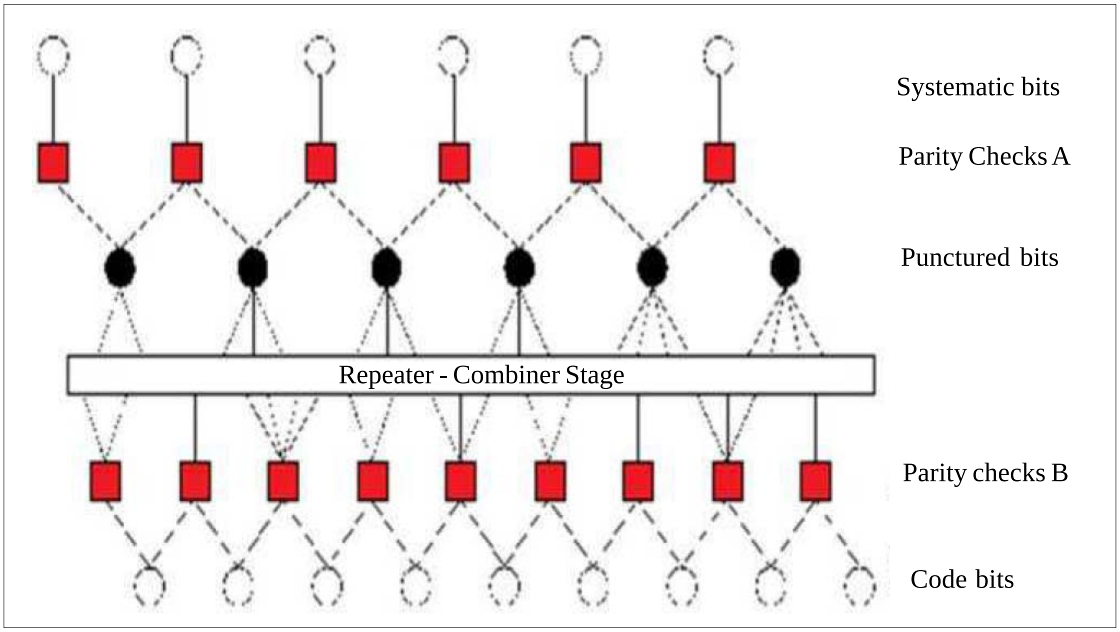


Figure 2. The Tanner graph based SC-ARA codes

3. Spatially Coupled Ara Codes

In this section we derive a closed form expression for the density evolution of a spatially coupled proto-graph ARA code ensemble on a BEC via message passing algorithm.

Consider a Tanner graph based (q, a, L, w) ARA ensemble, with a randomized parameter w, as shown in Figure 2, where we modify the construction as follows:

The set of "*punctured bit*" nodes is copied q times, interleaved in a spatially coupled form and added modulo-2 in a set of a bits to produce "*parity check B*" nodes.

We also introduce a smoothing randomized parameter w similar to that in [12]. However, the "*parity check B*" nodes are considered to be located at all integer positions $[-\infty, \infty]$ along with the extended $2L + 1$ coupled chain proto-graphs, but only "*parity check B*" nodes at position within the interval $[0, L + w - 2]$ actually interact with the "*punctured bit*" nodes to further the "*symbol bit*" nodes.

Let l denotes the iteration number. Referring to Figure 2, let x_a and x_f denote the probability of erasure messages from the "*parity check A*" nodes to the "*punctured bit*" nodes and vice-versa, let x_b and x_e denote the probability of erasure messages from the "*punctured bit*" nodes to the "*parity check B*" nodes and vice-versa, let x_c and x_d denote the probability of erasure messages from the "*parity check B*" nodes to the "*code bit*"
nodes and vice-versa.

The "*punctured bit*" nodes, the "*parity check B*" nodes and their interconnecting edges form the interlaced "*repeater-combiner*" stage of an ARA code. We are interested in spatially coupled the "*repeater-combiner*" stage, so that ARA code ensembles may inherit many properties.

Without loss of generality for fixed q, a the marginal density evolution equations in [6] are modified to be

$$x_{a_i}^{(l)} = 1 - (1 - x_{f_i}^{(l-1)})(1 - p)$$

$$x_{b_i}^{(l)} = (x_{a_i}^{(l)})^2 \left(\frac{1}{w} \sum_{j=1}^{w-1} x_{e_{i-j}}^{(l-1)} \right)^{a-1}$$

$$x_{c_i}^{(l)} = 1 - \left(\frac{1}{w} \sum_{j=1}^{w-1} 1 - x_{b_{i-j}}^{(l)} \right)^{q} (1 - x_{d_i}^{(l-1)})$$

$$x_{d_i}^{(l)} = p x_{c_i}^{(l)}$$

$$x_{e_i}^{(l)} = 1 - \left(\frac{1}{w} \sum_{j=0}^{w-1} 1 - x_{b_{i-j}}^{(l)} \right)^{q-1} (1 - x_{d_i}^{(l-1)})^2$$

$$x_{f_i}^{(l)} = (x_{a_i}^{(l)}) \left(\frac{1}{w} \sum_{i=1}^{w-1} x_{e_{i-j}}^{(l-1)} \right)^{a}$$

Where $x_{\beta_i}^{(l)}, \beta = a, b, c, d, e, f$ represent the fixed points for the extrinsic probability erasure messages between nodes, as previously investigated.

Now, we can solve for one of $x_{\beta_i}^{(l)}, \beta = a, b, c, d, e, f$ variables, these variables are arbitrary for $0 \le i \le (w-1)$ when plugging or substituting into the above set of equations.

Thereby, the density evolution of a proto-graph based (q, a, L, w) ensemble can be derived as

$$x_i = \frac{p^2 \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} \left[\frac{1-p}{1 - p\left(1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-k}\right)^{q}} \right]^2 \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-k} \right]^{q-1} \right]^{a-1}}{\left[1 - (1-p) \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} \left[\frac{1-p}{1 - p\left(1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-k}\right)^{q}} \right]^2 \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-l} \right]^{q-1} \right]^{a} \right]^2} \tag{7}$$

Likewise, if we apply the spatially coupling phenomenon in the design of the ensemble of non-systematic irregular repeat-accumulate (NSIRA) codes [14], then, the DE equation (14) in [14] will be modified as

$$x = \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} \left[\frac{1-p}{1 - p\left(1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-k}\right)^{q}} \right]^2 \left[1 - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j-l} \right]^{q-1} \right]^{a-1} \tag{8}$$

Following a similar approach to that is used for fixed q, a, in general, when ARA ensembles are characterized by varying order pairs of degree distributions, i.e. $L(x_i)$, $R(x_i)$, $\lambda(x_i)$ and $\rho(x_i)$, for fixed point variable x_i, then, the DE equation will be straight-forward as (4) except simply the fixed point variable x_i should be replaced by $\frac{1}{w} \sum_{j=0}^{w-1} x_{i-j}$

For decoding process to finish, the fixed point at $x_i = 0$ must be stable and to get decoding process started, the fixed point at $x_i = 1$ must be unstable. So, for fixed $q \ge 3$, $a \ge 3$, the ARA ensembles are unconditionally stable at $x_i = 0$ but the decoding chain reaction may fall in fast ending.

4. Capacity Achieving Spatially Coupled Ara Ensembles

In this section and the next section we will interpret a positive effect on the performance of ARA codes by imposing the spatially coupled structure on their “*repeater-combiner*” stage.

We will restrict attention to the case of randomized SC-ARA codes, with a randomized parameter w, of sequences of regular degree distributions that can achieve a threshold improvement over the BEC with *bounded density* per information bit. The *bounded density* per information bit of (q, a, L, w) SC-ARA ensembles can be investigated as follows :

First, the information bits are pre-coded with a rate-1 accumulator and finally, the parity bits are computed at the output of a second rate-1 accumulator.

Consider we have a coupled chain of $2L + 1$ proto-graphs, then, we have $2L + 1$ variable nodes per one proto-graph.

Equivalently, the check nodes are considered to be located at $[-\infty, \infty]$ and there are $\frac{q}{a}$ check nodes per one proto-gragh.

We assume that each of the a connections of the “*parity check B*” nodes at position i actually interact with the “*punctured bit*” nodes within the interval $[i - w + 1, i]$, and each of the q connections of the “*punctured bit*” nodes at position i actually interact with the “*parity check B*” nodes within the interval $[i, i + w - 1]$ for $-L \le i \le L$.

Let μ be The number of “*punctured bit*” nodes at position i, then, there are $\frac{q}{a}\ \mu$ “*parity check B*” nodes at the equivalent position.

There are $\frac{q}{a}\ \mu\,(2L - w)$ “*parity check B*” nodes actually interact with the “*punctured bit*” nodes inside the interval $[-L, L]$.

At the boundary $-L$ there are $\frac{q}{a}\mu\,(\sum_{i=0}^{w-1} 1 - (\frac{i}{w})^{a}\,)]$ average number of “*parity check B*” nodes actually interact with the “*punctured bit*” nodes inside the interval $[-L, L]$ and so is at the boundary L.

So, we have $\frac{q}{a}\,\mu\,[(2L - w) + 2\,(\sum_{i=0}^{w-1} 1 - (\frac{i}{w})^{a}\,)]$ average number of parity check nodes and $\mu\,[(2L + 1)$ number of variable nodes.

In the limit of large L the design coding rate is given as

$$R\,(q, a, L, w) = \frac{2L + 1}{2L + 1 + \frac{q}{a}\left(2L - w + 2\left(\sum_{i=0}^{w-1} 1 - (\frac{i}{w})^{a}\right)\right)} = \frac{1}{1 + \frac{q}{a}}\,, L \to \infty \tag{9}$$

The density is given by

$$D\,(q, a, L, w) = \frac{q\,(2L + 1)}{R\,(q, a, L, w)\,(2L + 1)} = q\left(1 + \frac{q}{a}\right), L \to \infty \tag{10}$$

The necessary and sufficient condition that the bit erasure probability converges to zero as the block length tends to infinity is given by

$$x_i \succ \frac{p^{2}\left[1 - \frac{1}{w}\sum_{j=0}^{w-1}\left[\frac{1-p}{1 - p\left(1 - \frac{1}{w}\sum_{j=0}^{w-1} x_{i+j-k}\right)^{q}}\right]^{2}\left[1 - \frac{1}{w}\sum_{j=0}^{w-1} x_{i+j-l}\right]^{q-1}\right]^{a-1}}{\left[1 - (1-p)\left[1 - \frac{1}{w}\sum_{j=0}^{w-1}\left[\frac{1-p}{1 - p\left(1 - \frac{1}{w}\sum_{j=0}^{w-1} x_{i+j-k}\right)^{q}}\right]^{2}\left[1 - \frac{1}{w}\sum_{j=0}^{w-1} x_{i+j-k}\right]^{q-1}\right]^{a}\right]^{2}} \tag{11}$$

This condition determines the maximum (threshold) value of the channel erasure probability for the non-trivial fixed point system,. i.e. ($\underline{x} := x_i, -L \le i \le L$), $\underline{x} \ne \underline{0}$, $p \in [0, 1]$, of successful decoding.

Let $\underline{x}$ be a fixed point system, then the average message entropy of x, i.e. Entro ($\underline{x}$) is defined as [12]

$$Entro\,(\underline{x}) = \frac{1}{2L+1} \sum_{i=-L}^{L} x_i \tag{12}$$

In the next section a procedure of establishing the existence of special fixed points for many different entropy values is repeated to produce a very useful curve, i.e. the extended EXIT curve, as a projected plot of these fixed points.

The spatial coupling idea is exploited for ARA ensembles to gain in convergence thresholds than conventional ARA ensembles.

After a predetermined maximum number of iterations, the fixed point system values of $\underline{x}$, substituted in (11), collapse for all $p < p_{threshold}$ where $p_{threshold}$ is the message-passing BP decoding threshold closed to its underlying MAP threshold and to the Shannon limit over the BEC.

Equation (7) can be rewritten as

$$\tilde{\lambda}\,(1 - \tilde{\rho}\;(1 - x_i)) = x_i \tag{13}$$

Where the tilted degree distributions $\tilde{\lambda}$ and $\tilde{\rho}$ are given by

$$\tilde{\lambda}\,(x_i) = \frac{p^{\,2}\left(\frac{1}{w}\left(\sum_{j=0}^{w-1} x_{i-j}\right)\right)^{a-1}}{\left[1-(1-p)\left(\frac{1}{w}\sum_{j=0}^{w-1} x_{i-j}\right)^{a}\right]^2} \tag{14}$$

$$\tilde{\rho}\,(x_i) = \frac{(1-p)^2\left(\frac{1}{w}\left(\sum_{j=0}^{w-1} x_{i-j}\right)\right)^{q-1}}{\left[(1-p)\left(\frac{1}{w}\sum_{j=0}^{w-1} x_{i-j}\right)^{q}\right]^2} \tag{15}$$

and represent new degree distributions after the graph reduction method [6] for the "*punctured bit*" nodes and the "*parity check B*" nodes, respectively.

A nice symmetry between information and parity bits can be obtained by swapping ρ with $1 - \rho$.

In general, the original degree distribution pair ($\lambda\,(x_i), \rho\,(x_i)$) (i.e., the original pair before the graph reduction) can be expressed in terms of the ($\lambda\,(x_i), -\tilde{\rho}\,(x_i)$) pair after some calculus as

$$\lambda\,(x_i) = \frac{\tilde{\lambda}\,(x_i)}{\left[p+(1-p)\left(\frac{\int_0^{x_i}\tilde{\lambda}\,(t)\,dt}{\int_0^{1}\tilde{\lambda}\,(t)\,dt}\right)\right]^2} \tag{16}$$

$$\rho\,(x_i) = \frac{\tilde{\rho}\,(x_i)}{\left[1-p+p\left(\frac{\int_0^{x_i}\tilde{\rho}\,(t)\,dt}{\int_0^{1}\tilde{\rho}\,(t)\,dt}\right)\right]^2} \tag{17}$$

such that $\tilde{L}(x_i = 1) = \left. \frac{\int_0^{x_i} \tilde{\lambda}(t)\, dt}{\int_0^1 \tilde{\lambda}(t)\, dt} \right|_{x_i = 1} = 1$ and $\tilde{R}(x_i = 1) = \left. \frac{\int_0^{x_i} \tilde{\rho}(t)\, dt}{\int_0^1 \tilde{\rho}(t)\, dt} \right|_{x_i = 1} = 1$

To obtain capacity achieving ARA ensembles with *bounded complexity* per information bit of a candidate degree distribution pair ($\lambda(x_i)$, $\rho(x_i)$), given a bounded distribution $\lambda(x_i)$ (here, a regular distribution), after calculating $\tilde{\lambda}(x_i)$ from (16), an algorithm to solve for $\tilde{\rho}(x_i)$ in terms of $\tilde{\lambda}(x_i)$ (or doing the inverse via symmetry property) for a certain ρ can be established following a similar robust work as in [6].

This algorithm tests the non-negativity of the resulting power series expansions of $\tilde{\rho}(x_i)$ and its corresponding original $\rho(x_i)$ around zero and determines the threshold value of ρ that can be used to construct an ensemble of capacity achieving.

If this is indeed the case, a *bounded complexity* per information bit is obtained by truncating the degree distribution $\rho(x_i)$ (or both $\lambda(x_i)$ and $\rho(x_i)$) such that the effect of each truncation is negligible as the distribution $\rho(x_i)$ has powers go to infinity (for a regular distribution $\rho(x_i)$, the power of the maximum value of relative weight in $\rho(x_i)$ is selected to be the value of q).

5. Simulation Results

In this section, we demonstrate the decoding performance of randomly SC-ARA ensembles and compare them at finite lengths with simulations of ARA ensembles constructed from self-matched LDPC codes [6].

We also draw the extended message-passing BP EXIT curve for the coupled ensemble, we apply (7) to each section i, $-L \leq i \leq L$ taking into account of the *spatial* structure of the code.

We then compute the extrinsic symbol estimate for each section and the corresponding message-passing extrinsic bit entropy. The extended message-passing BP EXIT curve of the SC ensemble is finally obtained by averaging over the $2L$ entropies of the chain.

Consider a comparative example of rate $\frac{1}{2}$ $(\bar{q}, \bar{a} = \frac{\bar{q}}{2}, 16, w)$ ensembles of block lengths 8184 with spatially coupling and 8192 without, respectively, where $\bar{q}$ represents the average weight distribution of $q(x)$ and we use $\bar{q} = 8$ for regular distribution.

In Figure 3, we plot the extended message-passing BP EXIT curve of the SC (8, 4, L, $w = L$) ensembles, for different values of L.

It follows that the message-passing BP threshold point $\rho_{threshold}$ is the point at which the left-most cliff edge of the curve vertically drops and for a large value of L it is very close to ½. However, there exists a small gap.

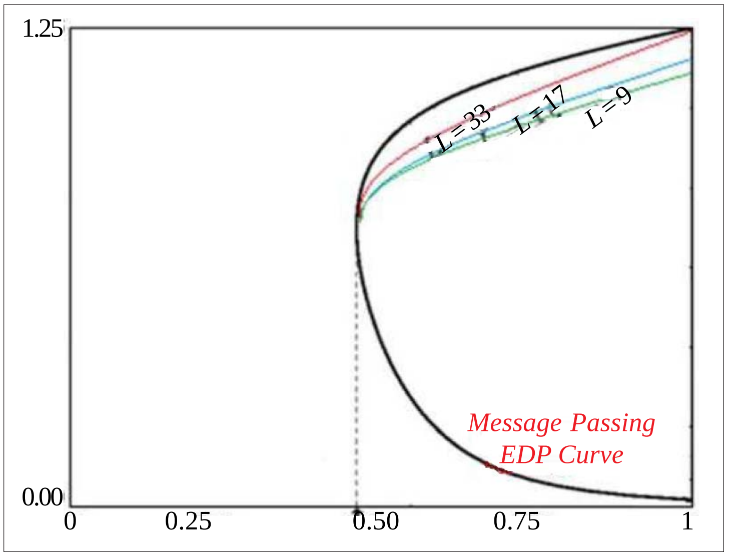
At the cliff edge there are some wiggles. The wiggle size decreases by increasing the randomized parameter w.

In Figure 4, we show the erasure decoding performance of (8, 4, 16, L, $w = L$) SC-ARA ensemble of length 8184 which requires a slightly higher rate than the self-matched ARA ensemble of length 8192 [6].

It is observed from the figure that the SC-ARA ensemble exhibits a better performance than the self-matched one especially throughout the waterfall region and as well as a better threshold.

It can be shown that SC-ARA ensemble can achieve a similar decoding threshold as the self-matched one with a higher rate.

Since the ensemble averaged performance (the entropy convergence performance) is simulated, high-rate outer codes are used to avoid cycles of length four and to lower the severe error floor due to small stopping sets. These outer codes are chosen uniformly at random from the ensemble of the binary linear block codes and their rate loss is neglected (rate-1codes).



$$p_{threshold} \cong p_{threshold}^{MAP}$$

Figure 3. The extended message-passing BP EXIT curve of the SC (8, 4, L, $w = L$) ensembles, for different values of L

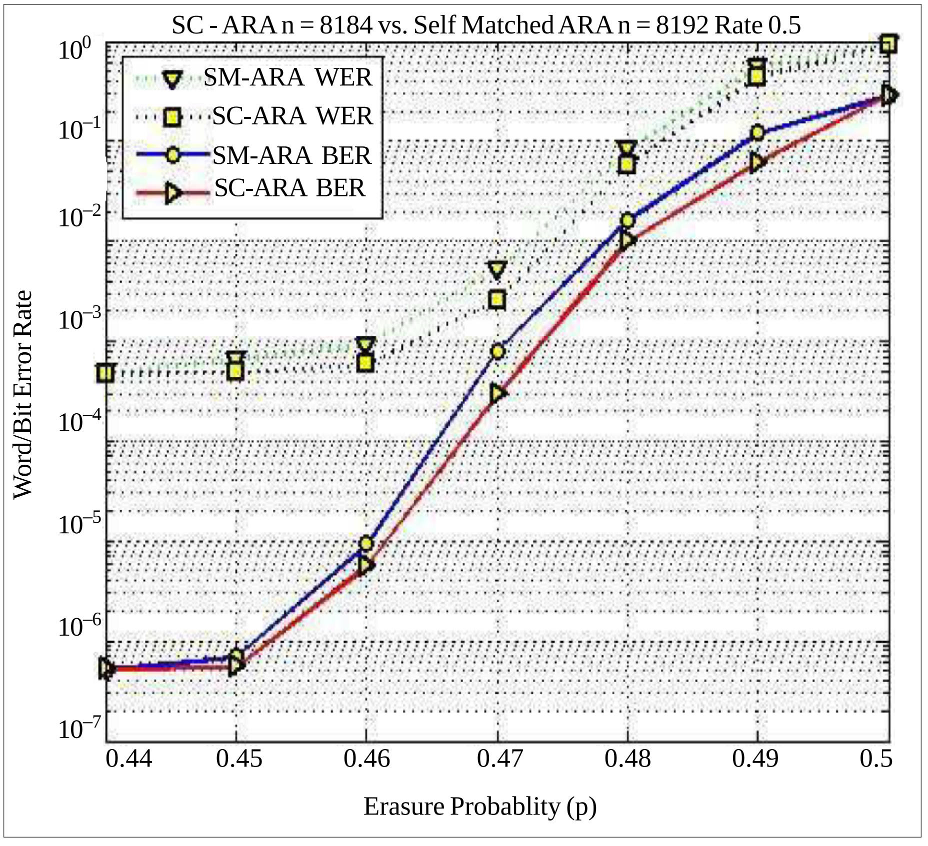


Figure 4. Performance comparison of (8, 4, 16, $w = 16$) SC-ARA ensemble of length 8184 and the self-matched ARA ensemble of length 8192 [6]

The complexity of these schemes are significantly bounded by the proper selection in part of $\lambda(x)$, $\rho(x)$, a and q such that (7) is satisfied.

6. Conclusion

In this paper, we provided a closed form relation for the density evolution message-passing analysis of capacity-achieving spatially-coupled ARA ensembles with bounded density and complexity under the BEC.

We also introduced an equivalent low density parity check code scheme for a SC-ARA code via graph reduction method under the BEC.

We further investigated the extended message-passing BP EXIT curve for the coupled ensemble. There exists a very small gap between message-passing BP threshold and the Shannon limit caused by wiggles.

Simulation results show that the performance of SC-ARA ensembles outperform the self-matched ARA ones in [6].

References

[1] Luby, M. G., Mitzenmacher, M., Shokrollahi, M. A., Spielman, D. A. (2001). Efficient erasure correcting codes, *IEEE Trans. on Information Theory*, 47, p. 569–584, February.

[2] Shokrollahi, M. A. (1999). New sequences of linear time erasure codes approaching the channel capacity, *In*: Proceedings of the 13th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Lectures Notes in Computer Science 1719, Springer Verlag, p. 65–76.

[3] Jin, H., McEliece, R. J. (2000). Irregular repeat-accumlate codes, *In*: Proceedings Second International Conference on Turbo Codes and Related Topics, p. 1–8, Brest, France, September.

[4] Khandekar, A., McEliece, R. J. (2001). On the complexity of reliable communication on the erasure channel, *In*: Proceedings 2001 IEEE International Symposium on Information Theory (ISIT2001), p.1, Washington, D. C., USA, June.

[5] Khandekar, A. (2002). Graph-based codes and iterative decoding, Ph.D. dissertation, California Institute of Technology, Pasadena, California, USA.

[6] Pfister, H., Sason, I. (2007). Accumulate repeat accumulate codes: Capacity-achieving ensembles of systematic codes for the erasure channel with bounded complexity, *IEEE Trans. Inf. Theory*, 53 (6) 2088 –2115, June.

[7] Jimenez Felstrom, A., Zigangirov, K. (1999). Time-varying periodic convolutional codes with low-density parity-check matrix, IEEE Trans. Inf. Theroy, 45 (6) 2181–2191, Sep.

[8] Sridharan, A., Lentmaier, M., Costello, D. J., Z. K. S. (2004). Convergence analysis of a class of LDPC convolutional codes for the erasure channel, in Allerton Conf. Commun., Control, and Computing, (Monticello, IL, USA), Oct, p. 953–962.

[9] Pusane, A., Feltstrom, A., Sridharan, A., Lentmaier, M., Zigangirov, K., Costello, D. (2008). Implementation aspects of LDPC convolutional codes, IEEE Trans. Commun., 56 (7) 1060–1069, Jul.

[10] Lentmaier, M., Sridharan, A., Costello, D., Zigangirov, K. (2000). Iterative decoding threshold analysis for LDPC convolutional codes, *IEEE Trans. Inf. Theroy*, 56 (10) 5274–5289, Oct.

[11] Lentmaier, M., Fettweis, G. P. (2010). On the thresholds of generalized LDPC convolutional codes based on protographs, in IEEE Int. Symposium on Inform. Theory, (Austin, USA), Jun. p. 709–713.

[12] Kudekar, S., Richardson, T., Urbanke, R. (2011). Threshold saturation via spatial coupling: Why convolutional LDPC ensembles erform so well over the BEC, *IEEE Trans. Inf. Theroy*, 57 (2) 803–834, Feb.

[13] Tazoe, K., Kasai, K., Sakaniwa, K. (2013). Efficient Termination of Spatially-Coupled Codes, arXiv:1302.1512v1 [cs.IT] 6 Feb.

[14] Jin, H., McEliece, R. J. (2000). Irregular repeat-accumlate codes, *In*: Proceedings Second International Conference on Turbo Codes and Related Topics, p. 1–8, Brest, France, September.