Time and Frequency Response Analysis of PID Controller

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ABSTRACT: *In this paper the control system is designed using Proportional-Integral-Differential (PID) controller with model transfer function of the plant in which temperature is to be controlled and an effort to present the practical circuitry and Time response and Frequency Response analysis is done both theoretically and by simulation. The response is testing by MatLab Simulink and PSpice for the same and a comparative result is finally presented.*

Keywords: Transfer Function, Time Response, Frequency Response, PID, Simulation

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1. Introduction

In this, the system is represented using block diagram using Simulink. Then the mathematical model is derived taking in consideration the entire control circuit Represented and tested on PSpice. Transfer function of PID controller is estimated using Z-N [1], [2], [6] and Routh's Stability criteria [1], [2], [6]. The time response analysis is carried out to calculate response parameters, frequency response and Stability Analysis is carried out theoretically and by simulation.

In this the system *refer Figure* 1 is represented as block diagram with Set point as the voltage required for maintaining and controlling desired temperature and unity feedback from the plant is used as input to the comparator and the 'error' is fed to the PID controller .Temperature dependent Plant model Transfer function is derived from combined transfer function of actuator and sensor circuit. As the control over available output voltage will be representing the control over the temperature. Main idea is verification of different parameters related to a control system and perform time and frequency response analysis.

2. Plant Model

The plant considered is having Heating process and a Thermometer system.

2.1 Heating Process

It consists of heating element; we also take in consideration the mass of tank m, heat transfer coefficient H_0, total area α and specific heat of the tank material C.

The transfer function of the heating process can be represented as [4].

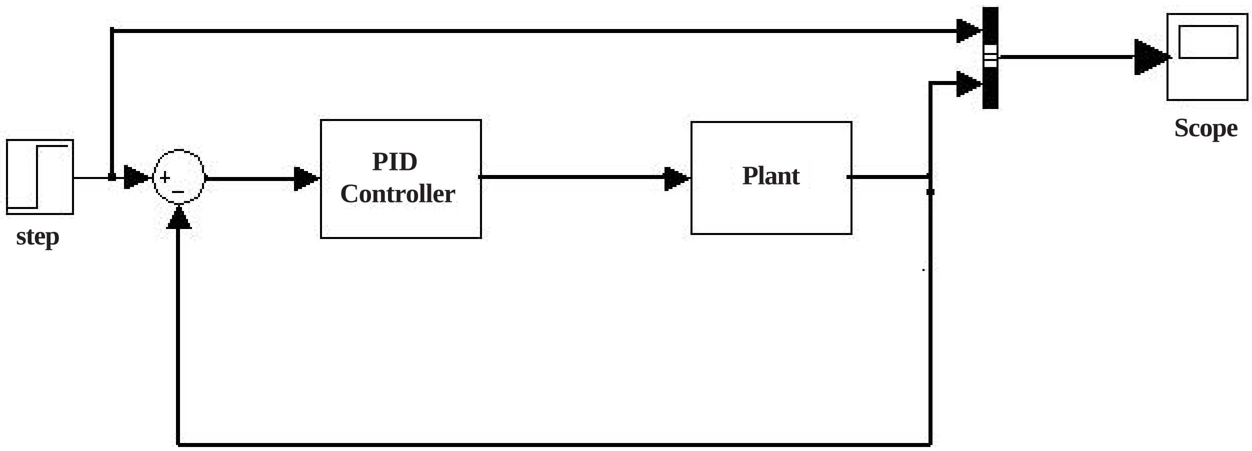

Figure 1. Block Diagram of System

$$T(s) = \frac{k}{s\tau + 1} \quad (1)$$

where k = $\frac{1}{\alpha H_0}$ & $\tau = \frac{mC}{\alpha H_0}$ and τ is time constant.

2.2 Temperature Sensor

In this we are considering a thermometer as sensor and its mathematical model is having a resistance connected in series with a capacitor and voltage across it changes with temperature.

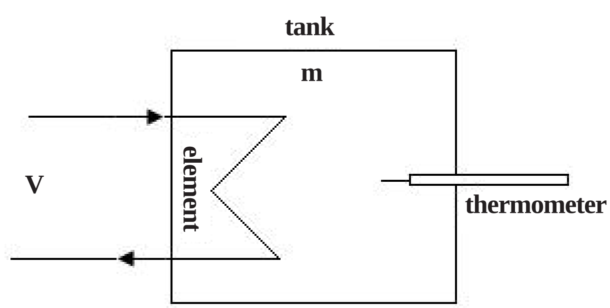

Figure 2. Plant Model

Then transfer function can be calculated by first applying Kirchoff's Law and finding *ei* and *eo*

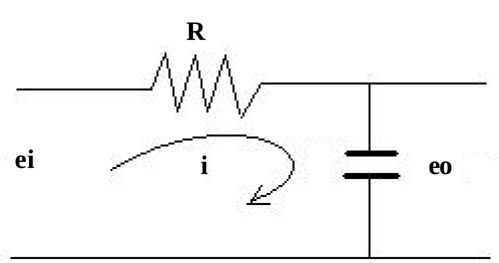

Figure 3. Electrical analog of Thermometer

$$Ri + \frac{1}{C}\int idt = ei \quad (2)$$

$$\frac{1}{C}\int idt = eo \tag{3}$$

Then by Laplace Transform of Equation 2 & Equation 3

$$RI(s) + \frac{1}{C}\frac{I(s)}{s} = Ei(s)$$

$$\frac{I(s)}{Cs} = Eo(s)$$

Now as $Eo(s) / Ei(s)$ represents the transfer function of Thermometer as [1],[2]

$$\frac{Eo(s)}{Ei(s)} = \frac{1}{RCs + 1} \tag{4}$$

Now as per the setup we are having the a single Plant so the combined transfer function can be represented as

$$\frac{k}{(s\tau + 1)(RCs + 1)} \tag{5}$$

From experimental data $RC = 1$ and $\tau = 0.2s$ the transfer function is

$$\frac{1}{s(0.4s^2 + 1.2s + 2)} \tag{6}$$

2.3 PID Controller

As modern control system is automated control system and most of the applications it is desirable to have auto – control instead of manual control. It is a feedback type control system with combination of **P**roportional, **I**ntegral & **D**erivative control over the system.

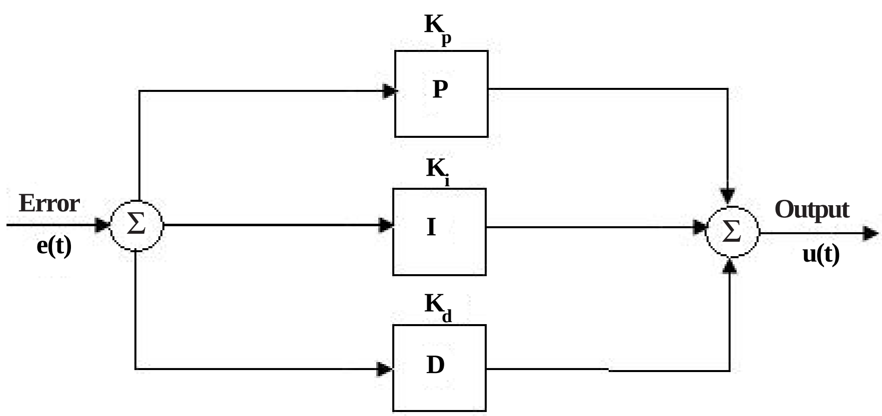

Figure 4. PID Controller

The proportional control has an important factor as proportional gain K_p, it amplifies the output proportional to the '*error*' signal received. In integral control the '*error*' is accumulated and multiplied by integral gain K_i. In derivative control the output changes at the rate of change of '*error*' with respect to time multiplied by derivative gain K_d.

In case of combined PID control the relation is represented as [1], [2], [6].

$$u(t) = K_p e(t) + K_i \int_0^t e(t)\, dt + K_d \frac{\partial e(t)}{\partial t} \tag{7}$$

By taking Laplace Transform and taking in consideration integral time T_i and derivative time T_d.

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \tag{8}$$

$$\text{where } K_i = \frac{K_p}{T_i} \ \& \ K_d = K_p T_d$$

From *Equation* 8 the transfer function of PID controller can be represented as

$$\frac{U(s)}{E(s)} = \frac{(T_d T_i s^2 + T_i s + 1) K_p}{T_i s} \quad (9)$$

2.4 Transfer Function of PID

To find the transfer function of PID controller we use Ziegler-Nichols rules and Routh's criteria [6]. So as to find the values of K_p, T_i, T_d.

In this we will use Ziegler-Nichols rules second method [1] in which we first set $T_i = \infty$ and $T_d = 0$ then use only proportional control K_p by increasing it from 0 to a critical value K_{cr} at which we get sustained oscillations to get corresponding period P_{cr} then we calculate the values as

Type of Controller	K_p	T_i	T_d
PID	$0.6 K_{cr}$	$0.6 P_{cr}$	$0.125 P_{cr}$

Table 1. Ziegler-Nichols rules [1][2][6][7]

The close loop transfer function after applying Ziegler-Nichols rules will be *refer Figure* 5

$$\frac{C(s)}{R(s)} = \frac{K_p}{s^3 + 3s^2 + 5s + K_p} \quad (10)$$

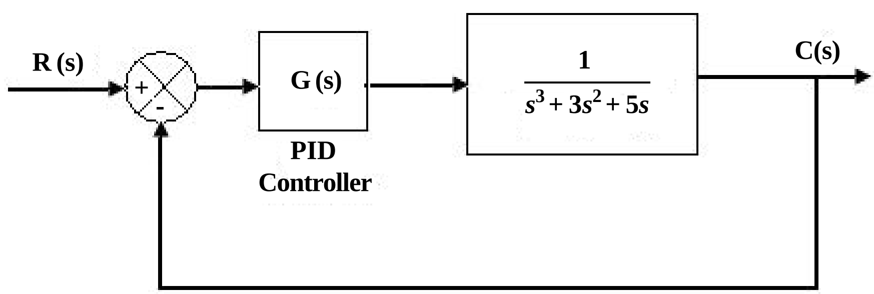

Figure 5. PID controlled system

The value of K_p that will make system marginally stable for sustained oscillations can be obtained by Routh stability criteria.The characteristic equation for close loop system is

$$s^3 + 3s^2 + 5s + K_p = 0 \quad (11)$$

So the Routh array becomes

s3	1	5
s2	3	K_p
s1	$\frac{15 - K_p}{3}$	
s0	K_p	

By examining the first column we observe that sustained oscillations will occur if $K_p = 15$, thus the critical gain K_{cr} is

$$K_{cr} = 15$$

Then we make proportional gain equal to critical gain to get the characteristic equation as from *Equation* 11

$$s^3 + 3s^2 + 5s + 15 = 0 \qquad (12)$$

Then frequency of sustained oscillations can be calculated by replacing $s = j\omega$ in *Equation* 12

$$(j\omega)^3 + 3(j\omega)^2 + 5(j\omega) + 15 = 0 \Rightarrow j\omega(j\omega)^2 - 3(j\omega)^2 + 5(j\omega) + 15 = 0$$

Or it can be written as

$$3(5 - \omega^2) + j\omega(5 - \omega^2) = 0 \qquad (13)$$

So the frequency for sustained oscillation will be $\omega^2 = 5$ or $\omega = \sqrt{5}$, and period will be

$$P_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}} = 2.8099 \qquad (14)$$

Then referring Table.1

$$K_p = 0.6\, K_{cr} = 9$$
$$T_i = 0.5\, P_{cr} = 1.40459$$
$$T_d = 0.125\, P_{cr} = 0.3512$$

Referring Equation 9 we can find transfer function of PID controller as

$$\frac{U(s)}{E(s)} = \frac{3.16s^2 + 8.99s + 6.41}{s} \qquad (15)$$

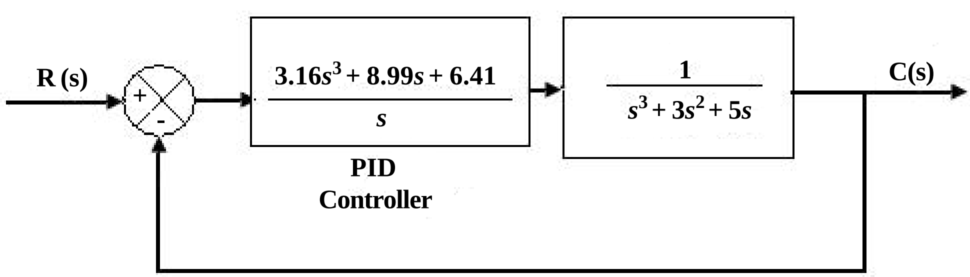

Figure 6. System with PID controller designed

So the closed loop transfer function of system will be

For $K_p = 9$

$$\frac{C(s)}{R(s)} = \frac{3.16s^2 + 8.99s + 6.41}{s^4 + 3s^3 + 8.16s^2 + 8.99s + 6.41} \qquad (16)$$

For $K_p = 18$

$$\frac{C(s)}{R(s)} = \frac{6.32s^2 + 18s + 12.82}{s^4 + 3s^3 + 11.32s^2 + 18s + 12.82} \qquad (17)$$

For $K_p = 9, T_i = 2.25, T_d = 0.5625$

$$\frac{C(s)}{R(s)} = \frac{5.06s^2 + 9s + 4}{s^4 + 3s^3 + 10.06s^2 + 9s + 4} \qquad (18)$$

Unit step response of the above system is

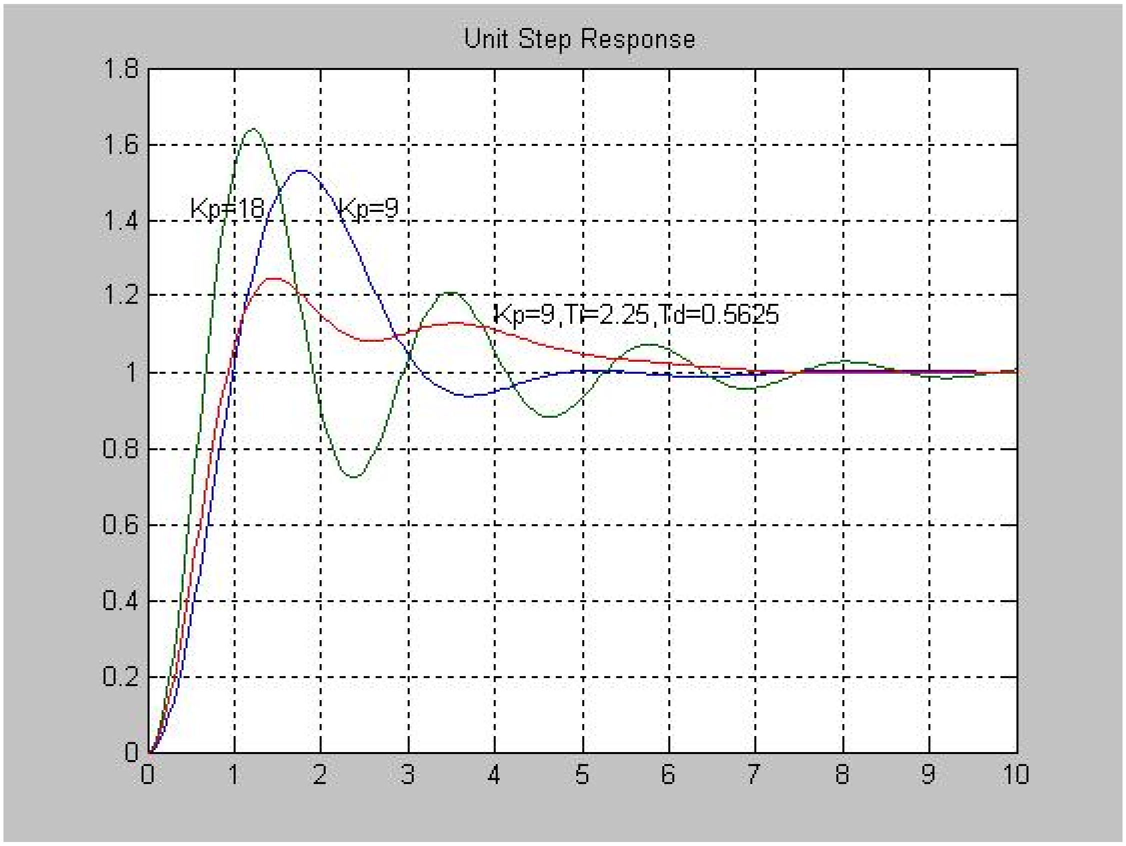

Figure 7. Unit step response of system represented in Equation16, 17, 18

3. Time Response Analysis

Time response comprises of transient response and steady state response. Transient response is which goes from initial to final state and steady state response is refers to behavior of output of system from $t \rightarrow \infty$[1]

3.1 Transient Response

A transient response is the out characteristic of a system, it generally exhibits damped oscillations before reaching steady state. In a transient response following parameters are considered.[1][2]

a.) Delay Time (t_d): It is the time required for the response to reach half the final value for first time.

b.) Rise Time (t_r): It is the time required by the response to reach to its final or 90% value from 0 to 10%.

c.) Peak Time (t_p): It is the time required by the response to reach to first peak of the overshoot.

d.) Maximum Overshoot (M_p): It is the maximum peak value reached reference to the unity value.

e.) Settling Time (t_s): It is the time required by the response to reach to become stable within an allowable range.

Let us consider Transfer function represented by *Equation* 16.

Then we will perform transient analysis [6] From the *Figure* 8 first of all we will find the damping ratio ζ and natural frequency ω_n.

We find that:

$$M_{p1} = 1.528 \; M_{p2} = 1.004$$
$$t_{p1} = 1.75 \; t_{p2} = 5.1 \; t_s = 9.15$$
$$y_\infty = 1.02 \; t_d = 0.63$$

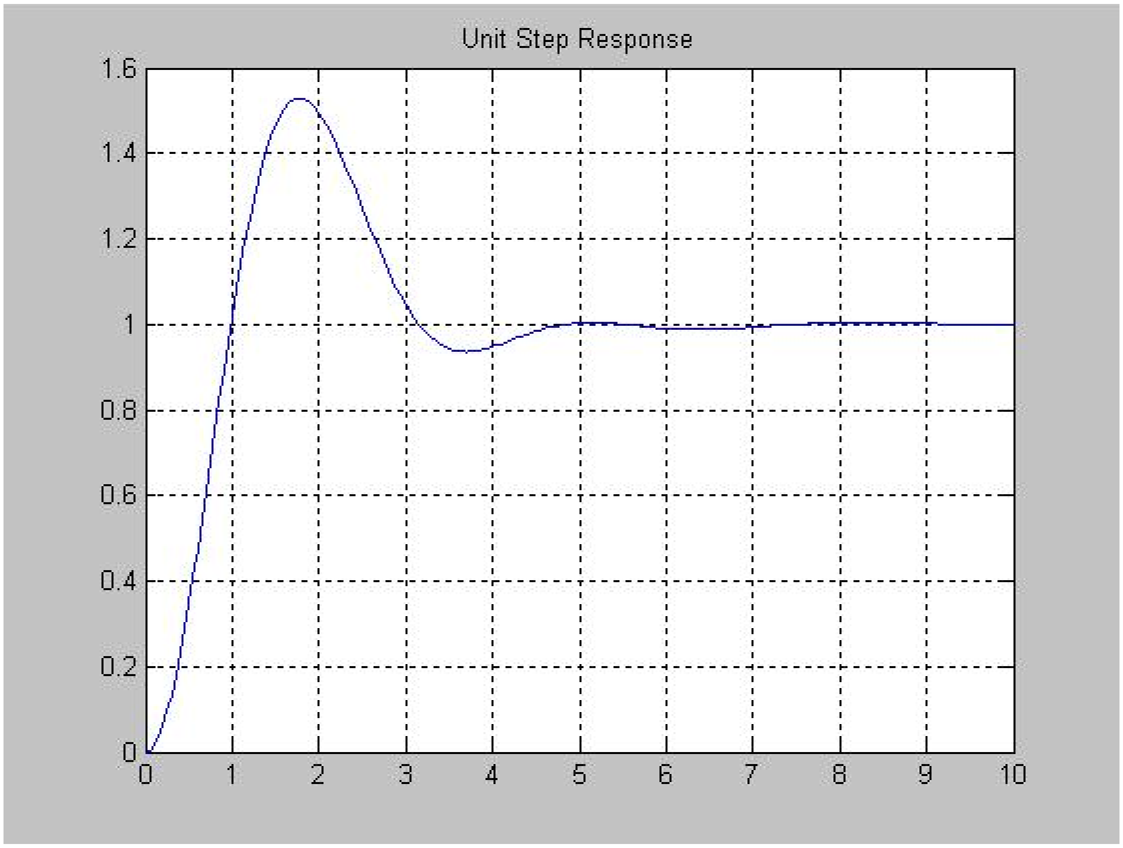

Figure 8. Unit step response of system represented in *Equation* 17

$$\zeta = \frac{1}{\sqrt{1\,(\frac{2\pi}{\ln d})^2}} \;\&\; \omega_n = \frac{2\pi}{(T_p\sqrt{1-\zeta^2})}$$

$$d = \frac{M_{p2} - y_\infty}{M_{p1} - y_\infty}$$

$$\Rightarrow \frac{1.004 - 1.02}{1.528 - 1.02} = -0.0314$$

$$T_p = t_{p2} - t_{p1}$$

$$\Rightarrow 5.1 - 1.75 = 3.35$$

So

$$\zeta = \frac{1}{\sqrt{1 + (\frac{2\pi}{\ln -0.0314})^2}} = 0.4828$$

$$\omega_n = \frac{2\pi}{(3.35\sqrt{1-(0.4828)^2})} = 2.143$$

Then we find the transfer function associated with the value of ζ and ω_n

$$\frac{C(s)}{R(s)} = \frac{4.5924}{s^2 + 2.0693s + 4.5924} \tag{19}$$

Output response for the above transfer function is

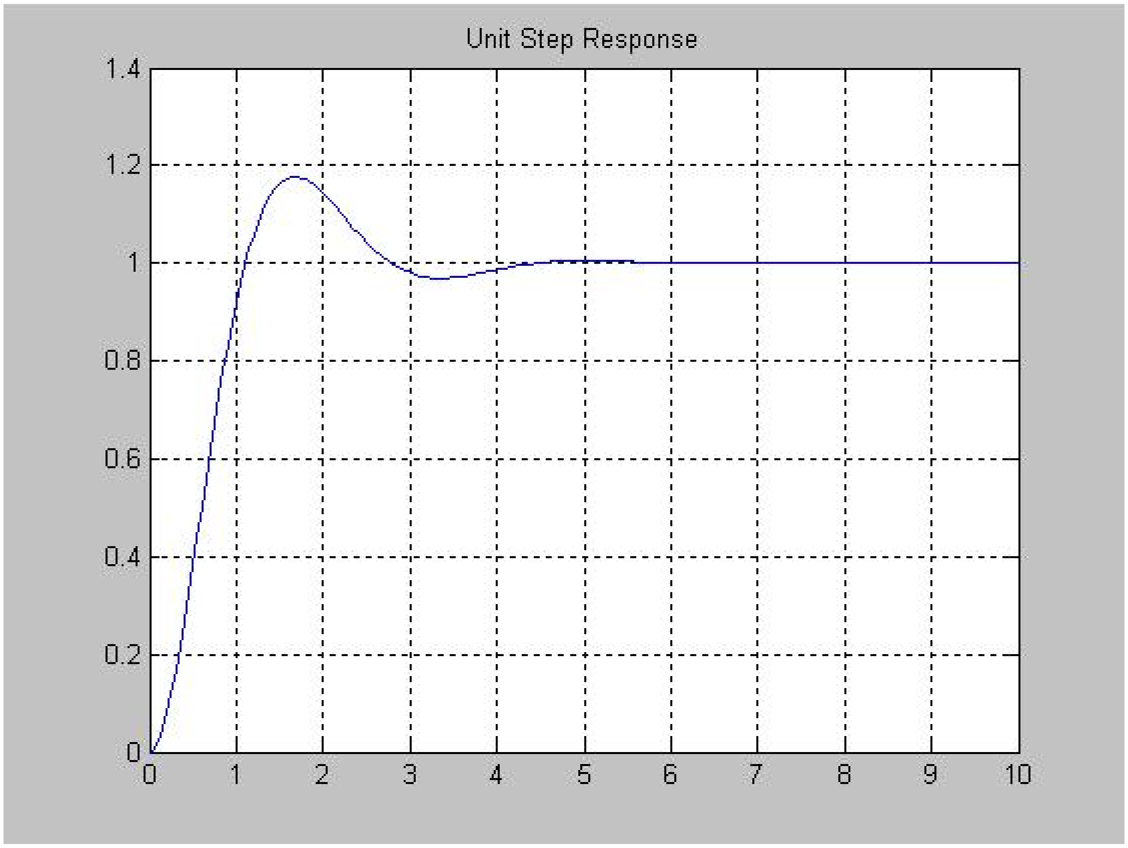

Figure 9. Unit step response of system represented in Equation 19

1) $Rise\ Time\ (t_r) = \dfrac{\pi - \phi}{\omega_d}$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \;\&\; \omega_d = \omega_n\sqrt{1-\zeta^2}$$

2) $Peek\ Time\ (t_p) = \dfrac{\pi}{\omega_d}$

3) $Maximunovershoot\ (M_p) = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

4) $Settling\ Time\ (t_s) = \dfrac{1}{\zeta\omega_n}$

$$\phi = \tan^{-1}\frac{\sqrt{1-(0.4828)^2}}{0.4828} = 1.066\ rad$$

$$\omega_d = 2.143\sqrt{1-(0.4828)^2} = 1.876\ rad/sec$$

1) $t_r = \frac{3.14 - 1.066}{1.876} = 1.10$ sec

2) $t_p = \frac{3.14}{1.876} = 1.73$ sec

3) $(M_p) = e^{-(0.4828\times3.14)/\sqrt{1-(0.4828)^2}} = 0.177$

4) $t_s = \frac{4}{0.4828\times 2.143} = 3.86$

According to the theoretical calculation the transient response parameters are as above and according to simulation are and according to simulation are

rise_time = 1.1500, peak_time = 1.6500
max_overshoot = 0.1767
settling_time = 3.8000

3.2 Steady State response

Now as can observe that control system is *Type*-1and the steady state error for unit step input for *Type*-1 is equal to zero[1][2] so $e_{ss} = 0$.

4. Frequency Response Analysis

Frequency response analysis refers to response of the system to input sine function.[1][2] We will use two methods for representing Frequency response analysis, first is '*Bode Plot*' which is same as step response in time domain and other is '*Nyquist Plot*' which will through light on absolute and relative stability analysis.

4.1 Bode Plot Analysis

It is graphical representation of system transfer function in frequency area. It has two plots magnitude plot and phase plot. For drawing first we get the open loop transfer function in frequency domain by replacing s by $j\omega$. So we have $G(j\omega)$ from *Equation.* 19 as

$$G(j\omega) = \frac{4.5924}{(j\omega)^2 + 2.0693\, j\omega} \quad (20)$$

1. Magnitude of resonance point (M_r). It is maximum value dimension in G($j\omega$). It is related to dimensions of overshoot and damping ratio ζ. $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

$$M_r = \frac{1}{2\times 0.4828\sqrt{1-(0.4828)^2}} = 1.18$$

It also expresses relative stability if $1.0 < M_r < 1.4$ i.e 0dB – 3dB, it corresponds to effective damping ratio $0.4 < \zeta < 0.7$ for satisfactory performance. If the value is greater than 1.5 then multiple overshoot will be available. [2]

2. Resonant Frequency (ω_r) it is the frequency at maximum resonant point and is related to natural frequency ω_n as $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ it indicates the speed of response.

$$\omega_r = 2.143\sqrt{1-2(0.4828)^2} = 1.565$$

3. Bandwidth (Bw) It indicates how the system traces the sine wave. Greater the bandwidth better the high frequency passes,

it is related to rise time and in general is proportional to response time. It is recommended that M_r should be small and ω_r should be high.[2]

$$Bw = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}}$$

$$x = \sqrt{2 - 4\,(0.4828)^2 + (0.4828)^4}$$

$$Bw = 2.143\sqrt{1 - 2\,(0.4828)^2 + x} \quad = 2.70$$

4. Gain & Phase Margin (*Gm* & *Pm*) Gain margin is difference between gain magnitude and 0dB at phase transit frequency ω_p. Phase margin is difference between the phase and -180° when gain is 0dB at gain transit frequency ω_g

$$Pm = 100 \times \zeta \Rightarrow 100 \times 0.4828 = 48.28$$

If the gain margin is negative below 0dB then it indicates stability. Grater the better.

These margins indicate the margin gain and phase has for system stability.

From simulation the values of M_r=1.1826, ω_r=1.519, Bw = 2.47, $Gm = \infty$, Pm = 50.42, $\omega_p = \infty$ & ω_g = 1.71 are Bode plot is

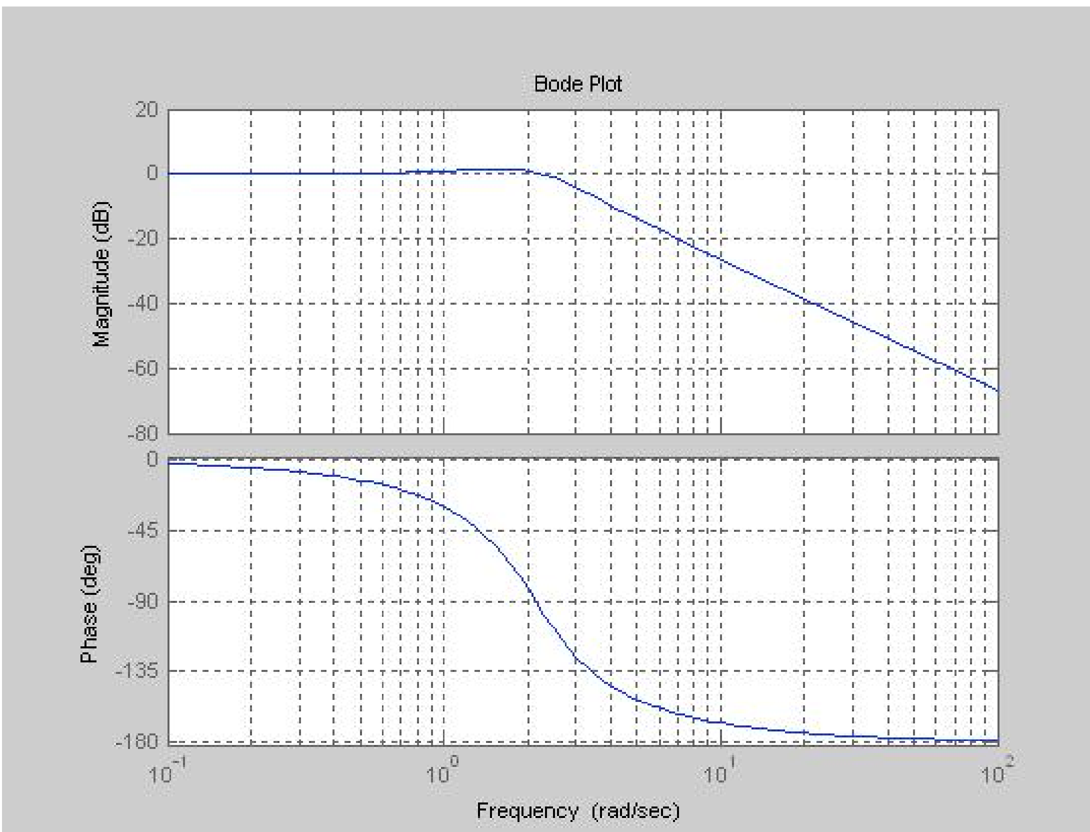

Figure 10. Bode Plot

4.2 Stability Analysis by Nyquist Plot

Nyquist stability criterion determines the stability of closed loop system from its open loop frequency response. It has two conditions for stability [1][2][4][5]

i) If the nyquist contour for open loop system does not encircle -1+ *j*0 point then the close loop system is stable.

ii) Along with the *i*) another condition is, there should be no poles to right of the *s* - plane

Let us consider the open loop transfer function represented by *Equation* 20. The Nyquist plot for the same is as

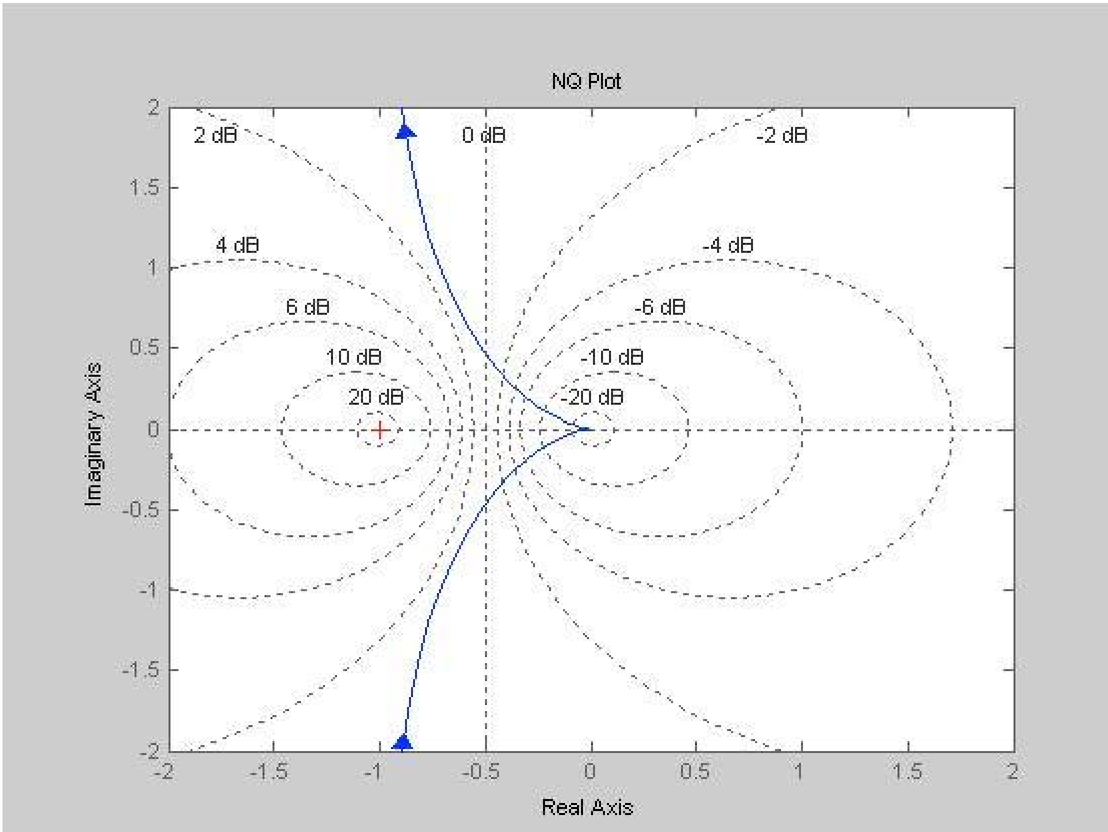

Figure 11. Nyquist plot for transfer function given by *Equation* 20

The Nyquist stability criterion can also be expressed as $Z = N + P$ where

Z = is number of zeros in right- half plane of *s*-plane.

N = is number of encircles of $-1+j0$

P = is number of Poles in right- half plane of *s*-plane.

As it is clear from the *Figure*11 that Z = 0, P = 0 & N = 0. So we can conclude that the system is stable.

Relative stability can also be analyzed; we observe that as the plot moves towards $-1+j0$ it tends towards instability. The relative stability can be measured in terms of a & ϕ. They refer to Gain margin and Phase margin. The value a refers to the point when the locus crosses the axis at 180° and the angle ϕ refers to the angle the locus forms with the axis. [2]

The gain margin *Gm* and phase margin *Pm* is calculated as

$$Gm = \left[\frac{KT_1T_2}{T_1 + T_2}\right]^{-1}$$

$$Pm = \tan^{-1}\left[2\zeta\left\{\sqrt{\frac{1}{\sqrt{(4\zeta^4+1)} - 2\zeta^2}}\right\}\right]$$

In case when $Gm = \infty$ then *Pm* is the correct measure of relative stability.

So we find Pm by putting the value of damping factor in the formula

Parameter	Time Response		Frequency Response	
	Cal.	Sim.	Cal.	Sim.
Damping ratio (ξ)	0.4828	The system is under damped		
Rise Time (t_r)	1.10s	1.15s	-	-
Peak Time (t_p)	1.73s	1.65s	-	-
Settling Time(t_s)	3.86s	3.80s	-	-
Max. Overshoot(M_p)	0.177	0.176	-	-
Max. magnitude (M_r)	-	-	1.18	1.18
Natural frequency(ω_n)	2.143		-	
Resonance frequency(ω_r)	-	-	1.565	1.519
Bandwidth(B_w)	-	-	2.70	2.47
Phase Margin(P_m)	-	-	48.28/50.19	50.42/53.99
Gain Margin(G_m)	-	-	-	∞
Cross over gain(ω_g)	-	-	-	1.71
Cross over phase(ω_p)	-	-	-	∞
Stability	-	-	Yes	
Relative Stability	-	-	Stable for all values of K < 4.6	

Table 2. Comparison of Calculated and simulated output

$$Pm = \tan^{-1}\left[2 \times 0.4828 \left\{ \sqrt{\frac{1}{\sqrt{x} - 2\times(0.4828)^2}} \right\}\right]$$

$$x = \sqrt{(4\times(0.4828)^4 + 1} = 1.1033$$

From the Figure 12 we can calculate Pm as

$$Pm = 50.19$$

$$Pm = \tan^{-1}\frac{0.809}{0.5878} = 53.99^{o}$$

This value of Pm varies with value of K gain. Value of Pm decreases with increasing K. For $K = 4.59$ $Pm = 53.99$, $K = 6$ $Pm = 36$, $K = 30$ $Pm = 18$

5. Conclusion

In this paper temperature control system using PID is derived mathematically and the Time response and Frequency response analysis is carried out theoretically and by simulation and different parameters are calculated.

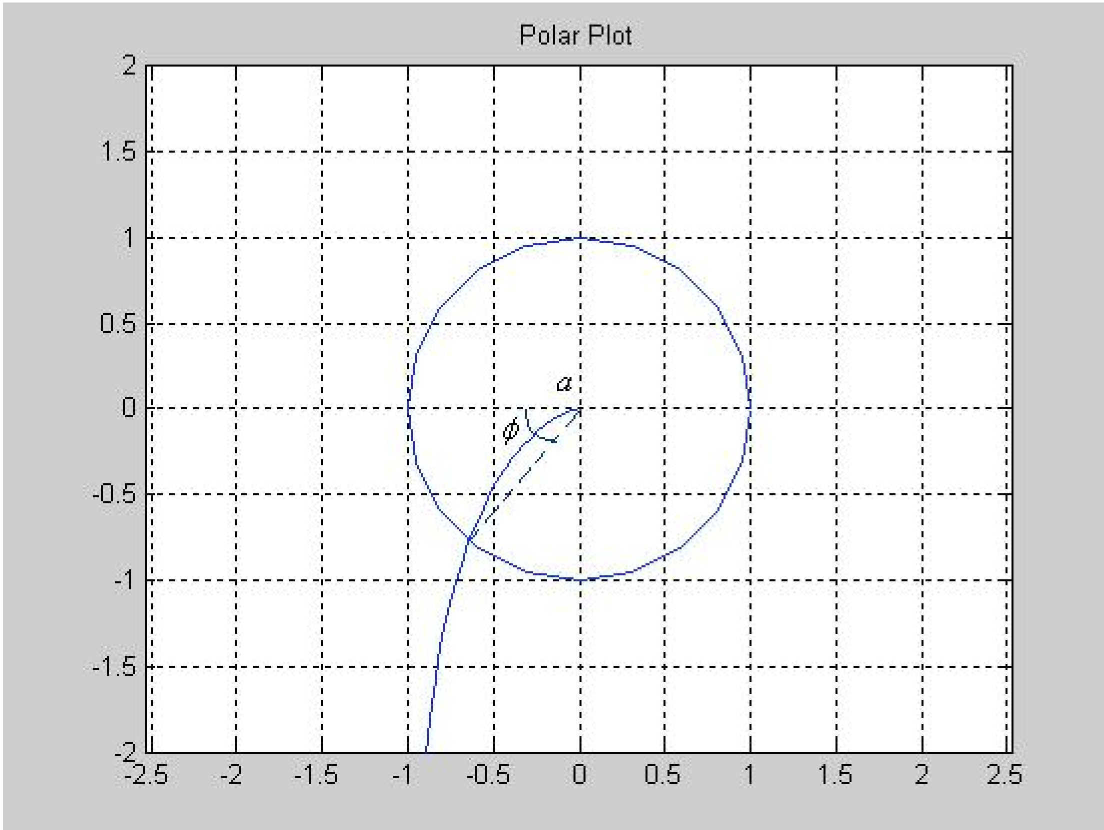

Figure 12. Polar Plot for transfer function Equation.21

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