Time and Frequency Response Analysis of PID Controller

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ABSTRACT: In this paper the control system is designed using Proportional-Integral-Differential (PID) controller with model transfer function of the plant in which temperature is to be controlled and an effort to present the practical circuitry and Time response and Frequency Response analysis is done both theoretically and by simulation. The response is testing by MatLab Simulink and PSpice for the same and a comparative result is finally presented.

Keywords: Transfer Function, Time Response, Frequency Response, PID, Simulation

Received: 10 March 2012, Revised 19 April 2012, Accepted 28 April 2012

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1. Introduction

In this, the system is represented using block diagram using Simulink. Then the mathematical model is derived taking in consideration the entire control circuit Represented and tested on PSpice. Transfer function of PID controller is estimated using Z-N [1], [2], [6] and Routh’s Stability criteria [1], [2], [6]. The time response analysis is carried out to calculate response parameters, frequency response and Stability Analysis is carried out theoretically and by simulation.

In this the system refer Figure 1 is represented as block diagram with Set point as the voltage required for maintaining and controlling desired temperature and unity feedback from the plant is used as input to the comparator and the ‘error’ is fed to the PID controller. Temperature dependent Plant model Transfer function is derived from combined transfer function of actuator and sensor circuit. As the control over available output voltage will be representing the control over the temperature. Main idea is verification of different parameters related to a control system and perform time and frequency response analysis.

2. Plant Model

The plant considered is having Heating process and a Thermometer system.

2.1 Heating Process

It consists of heating element; we also take in consideration the mass of tank \( m \), heat transfer coefficient \( H_0 \), total area \( \alpha \) and specific heat of the tank material \( C \).

The transfer function of the heating process can be represented as [4].
$T(s) = \frac{k}{s\tau + 1}$ \hspace{1cm} (1)

where $k = \frac{1}{\alpha H_0}$ & $\tau = \frac{mC}{\alpha H_0}$ and $\tau$ is time constant.

### 2.2 Temperature Sensor

In this we are considering a thermometer as sensor and its mathematical model is having a resistance connected in series with a capacitor and voltage across it changes with temperature.

Then transfer function can be calculated by first applying Kirchoff’s Law and finding $e_i$ and $e_o$

\[ Ri + \frac{1}{C} \int i dt = e_i \] \hspace{1cm} (2)
\[
\int \frac{1}{C} \text{idt} = e_0
\]

Then by Laplace Transform of Equation 2 & Equation 3

\[
Rl(s) + \frac{1}{C} \frac{I(s)}{s} = Ei(s)
\]

\[
I(s) = Eo(s)
\]

\[
\frac{I(s)}{Cs} = Eo(s)
\]

Now as \(Eo(s) / Ei(s)\) represents the transfer function of Thermometer as \([1],[2]\)

\[
Eo(s) = \frac{1}{RCS + 1} Ei(s)
\]

Now as per the setup we are having the a single Plant so the combined transfer function can be represented as

\[
\frac{k}{(s\tau + 1)(RCs + 1)}
\]

From experimental data \(RC = 1\) and \(\tau = 0.2\) the transfer function is

\[
\frac{1}{s(0.4s^2 + 1.2s + 2)}
\]

2.3 PID Controller

As modern control system is automated control system and most of the applications it is desirable to have auto – control instead of manual control. It is a feedback type control system with combination of Proportional, Integral & Derivative control over the system.

![PID Controller Diagram]

The proportional control has an important factor as proportional gain \(K_p\), it amplifies the output proportional to the ‘error’ signal received. In integral control the ‘error’ is accumulated and multiplied by integral gain \(K_i\). In derivative control the output changes at the rate of change of ‘error’ with respect to time multiplied by derivative gain \(K_d\).

In case of combined PID control the relation is represented as \([1],[2],[6]\).

\[
u(t) = K_p e(t) + K_1 \int_0^t e(t) dt + K_d \frac{\partial e(t)}{\partial t}
\]

By taking Laplace Transform and taking in consideration integral time \(T_i\) and derivative time \(T_d\)

\[
\frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right)
\]
where \( K_i = \frac{K_p}{T_i} \) & \( K_d = K_p T_d \).

From Equation 8 the transfer function of PID controller can be represented as

\[
\frac{U(s)}{E(s)} = \frac{(T_d T_i s^2 + T_i s + 1) K_p}{T_i s} \tag{9}
\]

2.4 Transfer Function of PID

To find the transfer function of PID controller we use Ziegler-Nichols rules and Routh’s criteria [6]. So as to find the values of \( K_p, T_i, T_d \).

In this we will use Ziegler-Nichols rules second method [1] in which we first set \( T_i = \infty \) and \( T_d = 0 \) then use only proportional control \( K_p \) by increasing it from 0 to a critical value \( K_{cr} \) at which we get sustained oscillations to get corresponding period \( P_{cr} \) then we calculate the values as

<table>
<thead>
<tr>
<th>Type of Controller</th>
<th>( K_p )</th>
<th>( T_i )</th>
<th>( T_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>0.6 ( K_{cr} )</td>
<td>0.6 ( P_{cr} )</td>
<td>0.125 ( P_{cr} )</td>
</tr>
</tbody>
</table>

Table 1. Ziegler-Nichols rules [1][2][6][7]

The close loop transfer function after applying Ziegler-Nichols rules will be refer Figure 5

\[
\frac{C(s)}{R(s)} = \frac{K_p}{s^3 + 3s^2 + 5s + K_p} \tag{10}
\]

![Figure 5. PID controlled system](image)

The value of \( K_p \) that will make system marginally stable for sustained oscillations can be obtained by Routh stability criteria. The characteristic equation for close loop system is

\[
s^3 + 3s^2 + 5s + K_p = 0 \tag{11}
\]

So the Routh array becomes

\[
\begin{array}{ccc}
s^3 & 1 & 5 \\
s^2 & 3 & K_p \\
s^1 & \frac{15 - K_p}{3} & K_p \\
s^0 & K_p & \\
\end{array}
\]

By examining the first column we observe that sustained oscillations will occur if \( K_p = 15 \), thus the critical gain \( K_{cr} \) is
\[ K_{cr} = 15 \]

Then we make proportional gain equal to critical gain to get the characteristic equation as from Equation 11

\[ s^3 + 3s^2 + 5s + 15 = 0 \]  \hspace{1cm} (12)

Then frequency of sustained oscillations can be calculated by replacing \( s = j\omega \) in Equation 12

\[(j\omega)^3 + 3(j\omega)^2 + 5(j\omega) + 15 = 0 \Rightarrow j\omega (j\omega)^2 - 3(j\omega)^2 + 5(j\omega) + 15 = 0\]

Or it can be written as

\[ 3(5 - \omega^2) + j\omega (5 - \omega^2) = 0 \]  \hspace{1cm} (13)

So the frequency for sustained oscillation will be \( \omega^2 = 5 \) or \( \omega = \sqrt{5} \), and period will be

\[ P_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}} = 2.8099 \]  \hspace{1cm} (14)

Then referring Table 1

\[ K_p = 0.6 \quad K_{cr} = 9 \]
\[ T_i = 0.5 \quad P_{cr} = 1.40459 \]
\[ T_d = 0.125 \quad P_{cr} = 0.3512 \]

Referring Equation 9 we can find transfer function of PID controller as

\[ \frac{U(s)}{E(s)} = \frac{3.16s^2 + 8.99s + 6.41}{s} \]  \hspace{1cm} (15)

![Figure 6. System with PID controller designed](image)

So the closed loop transfer function of system will be

For \( K_p = 9 \)

\[ \frac{C(s)}{R(s)} = \frac{3.16s^2 + 8.99s + 6.41}{s^4 + 3s^3 + 8.16s^2 + 8.99s + 6.41} \]  \hspace{1cm} (16)

For \( K_p = 18 \)

\[ \frac{C(s)}{R(s)} = \frac{6.32s^2 + 18s + 12.82}{s^4 + 3s^3 + 11.32s^2 + 18s + 12.82} \]  \hspace{1cm} (17)

For \( K_p = 9, T_i = 2.25, T_d = 0.5625 \)

\[ \frac{C(s)}{R(s)} = \frac{5.06s^2 + 9s + 4}{s^4 + 3s^3 + 10.06s^2 + 9s + 4} \]  \hspace{1cm} (18)

Unit step response of the above system is
3. Time Response Analysis

Time response comprises of transient response and steady state response. Transient response is which goes from initial to final state and steady state response is refers to behavior of output of system from $t \to \infty$ [1]

3.1 Transient Response

A transient response is the out characteristic of a system, it generally exhibits damped oscillations before reaching steady state. In a transient response following parameters are considered. [1][2]

a.) Delay Time ($t_d$): It is the time required for the response to reach half the final value for first time.

b.) Rise Time ($t_r$): It is the time required by the response to reach to its final or 90% value from 0 to 10%.

c.) Peak Time ($t_p$): It is the time required by the response to reach to first peak of the overshoot.

d.) Maximum Overshoot ($M_p$): It is the maximum peak value reached reference to the unity value.

e.) Settling Time ($t_s$): It is the time required by the response to reach to become stable within an allowable range.

Let us consider Transfer function represented by Equation 16.

Then we will perform transient analysis [6] From the Figure 8 first of all we will find the damping ratio $\zeta$ and natural frequency $\omega_n$.

We find that:

$$M_{p1} = 1.528, \quad M_{p2} = 1.004$$
$$t_{p1} = 1.75, \quad t_{p2} = 5.1, \quad t_s = 9.15$$
$$y_\infty = 1.02, \quad t_d = 0.63$$
Figure 8. Unit step response of system represented in Equation 17

\[ \zeta = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\ln d}\right)^2}} \quad \& \quad \omega_n = \frac{2\pi}{(T_p \sqrt{1 - \zeta^2})} \]

\[ d = \frac{M_p2 - y_m}{M_p1 - y_m} \]

\[ \Rightarrow \frac{1.004 - 1.02}{1.528 - 1.02} = -0.0314 \]

\[ T_p = t_{p2} - t_{p1} \]

\[ \Rightarrow 5.1 - 1.75 = 3.35 \]

So

\[ \zeta = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\ln 0.0314}\right)^2}} = 0.4828 \]

\[ \omega_n = \frac{2\pi}{(3.35 \sqrt{1 - (0.4828)^2})} = 2.143 \]
Then we find the transfer function associated with the value of $\zeta$ and $\omega_n$

$$\frac{C(s)}{R(s)} = \frac{4.5924}{s^2 + 2.0693s + 4.5924}$$  \hspace{1cm} (19)

Output response for the above transfer function is

![Unit Step Response](image)

**Figure 9. Unit step response of system represented in Equation 19**

1) **Rise Time** ($t_r$) = $\frac{\pi - \phi}{\omega_d}$

$$\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \quad \& \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}$$

2) **Peak Time** ($t_p$) = $\frac{\pi}{\omega_d}$

3) **Maximum overshoot** ($M_p$) = $e^{\zeta \pi / \sqrt{1 - \zeta^2}}$

4) **Settling Time** ($t_s$) = $\frac{1}{\zeta \omega_n}$

$$\omega_d = 2.143 \sqrt{1 - (0.4828)^2} = 1.876 \text{ rad/ sec}$$

$$\phi = \tan^{-1} \left( \frac{\sqrt{1 - (0.4828)^2}}{0.4828} \right) = 1.066 \text{ rad}$$
1) \[ t_r = \frac{3.14 - 1.066}{1.876} = 1.10 \text{ sec} \]

2) \[ t_p = \frac{3.14}{1.876} = 1.73 \text{ sec} \]

3) \[ (M_p) = e^{(0.4828 \times 3.14)} \sqrt{1 - (0.4828)^2} = 0.177 \]

4) \[ t_s = \frac{4}{0.4828 \times 2.143} = 3.86 \]

According to the theoretical calculation the transient response parameters are as above and according to simulation are

\begin{align*}
\text{rise time} &= 1.1500, \quad \text{peak time} = 1.6500 \\
\text{max overshoot} &= 0.1767 \\
\text{settling time} &= 3.8000
\end{align*}

3.2 Steady State response

Now as can observe that control system is Type-1 and the steady state error for unit step input for Type-1 is equal to zero[1][2] so \( e_{ss} = 0 \).

4. Frequency Response Analysis

Frequency response analysis refers to response of the system to input sine function.[1][2] We will use two methods for representing Frequency response analysis, first is ‘Bode Plot’ which is same as step response in time domain and other is ‘Nyquist Plot’ which will throw light on absolute and relative stability analysis.

4.1 Bode Plot Analysis

It is graphical representation of system transfer function in frequency area. It has two plots magnitude plot and phase plot. For drawing first we get the open loop transfer function in frequency domain by replacing \( s \) by \( j\omega \). So we have \( G(j\omega) \) from Equation. 19 as

\[ G(j\omega) = \frac{4.5924}{(j\omega)^2 + 2.0693 \cdot j\omega} \]

1. Magnitude of resonance point \( (M_r) \). It is maximum value dimension in \( G(j\omega) \). It is related to dimensions of overshoot and damping ratio \( \zeta \). \( M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \)

\[ M_r = \frac{1}{2 \times 0.4828 \sqrt{1 - (0.4828)^2}} = 1.18 \]

It also expresses relative stability if \( 1.0 < M_r < 1.4 \) i.e 0dB – 3dB, it corresponds to effective damping ratio \( 0.4 < \zeta < 0.7 \) for satisfactory performance. If the value is greater than 1.5 then multiple overshoot will be available. [2]

2. Resonant Frequency \( (\omega_r) \) it is the frequency at maximum resonant point and is related to natural frequency \( \omega_n \) as \( \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \) it indicates the speed of response.

\[ \omega_r = 2.143 \sqrt{1 - 2(0.4828)^2} = 1.565 \]

3. Bandwidth \( (B_\omega) \) It indicates how the system traces the sine wave. Greater the bandwidth better the high frequency passes,
it is related to rise time and in general is proportional to response time. It is recommended that $M_r$ should be small and $\omega_r$ should be high.[2]

$$Bw = \omega_n \sqrt{1 - 2 \zeta^2 + \sqrt{2 - 4 \zeta^2 + \zeta^4}}$$

$$\chi = \sqrt{2 - 4 (0.4828)^2 + (0.4828)^4}$$

$$Bw = 2.143 \frac{\sqrt{1 - 2 (0.4828)^2 + \chi}}{\chi} = 2.70$$

4. Gain & Phase Margin ($Gm$ & $Pm$) Gain margin is difference between gain magnitude and 0dB at phase transit frequency $\omega_p$. Phase margin is difference between the phase and -180° when gain is 0dB at gain transit frequency $\omega_g$

$$Pm = 100 \times \zeta \Rightarrow 100 \times 0.4828 = 48.28$$

If the gain margin is negative below 0dB then it indicates stability. Greater the better.

These margins indicate the margin gain and phase has for system stability.

From simulation the values of $M_r=1.1826$, $\omega_r=1.519$, $Bw=2.47$, $Gm=\infty$, $Pm=50.42$, $\omega_p=\infty$ & $\omega_g=1.71$ are Bode plot is

![Bode Plot](image)

Figure 10. Bode Plot

4.2 Stability Analysis by Nyquist Plot

Nyquist stability criterion determines the stability of closed loop system from its open loop frequency response. It has two conditions for stability [1][2][4][5]

i) If the nyquist contour for open loop system does not encircle $-1+j0$ point then the close loop system is stable.

ii) Along with the i) another condition is, there should be no poles to right of the $s$-plane

Let us consider the open loop transfer function represented by Equation 20. The Nyquist plot for the same is as
The Nyquist stability criterion can also be expressed as $Z = N + P$ where

$Z$ = number of zeros in right- half plane of $s$-plane.

$N$ = number of encircles of $-1 + j0$

$P$ = number of Poles in right- half plane of $s$-plane.

As it is clear from the Figure 11 that $Z = 0, P = 0 \& N = 0$. So we can conclude that the system is stable.

Relative stability can also be analyzed; we observe that as the plot moves towards $-1 + j0$ it tends towards instability. The relative stability can be measured in terms of $a$ & $\phi$. They refer to Gain margin and Phase margin. The value $a$ refers to the point when the locus crosses the axis at $180^\circ$ and the angle $\phi$ refers to the angle the locus forms with the axis. [2]

The gain margin $Gm$ and phase margin $Pm$ is calculated as

$$Gm = \left(\frac{KT_1T_2}{T_1 + T_2}\right)^{-1}$$

$$Pm = \tan^{-1}\left[2\zeta \sqrt{\frac{1}{(4\zeta^2 + 1) - 2\zeta^2}}\right]$$

In case when $Gm = \infty$ then $Pm$ is the correct measure of relative stability.

So we find Pm by putting the value of damping factor in the formula.

Figure 11. Nyquist plot for transfer function given by Equation 20.
\[ Pm = \tan^{-1} \left[ 2 \times 0.4828 \left\{ \sqrt{\frac{1}{\sqrt{x} - 2 \times (0.4828)^2}} \right\} \right] \]

\[ x = \sqrt{4 \times (0.4828)^2 + 1} = 1.1033 \]

From the Figure 12 we can calculate \( Pm \) as

\[ Pm = 50.19 \]

\[ Pm = \tan^{-1} \left( \frac{0.809}{0.5878} \right) = 53.99^\circ \]

This value of \( Pm \) varies with value of \( K \) gain. Value of \( Pm \) decreases with increasing \( K \). For \( K = 4.59 \) \( Pm = 53.99 \), \( K = 6 \) \( Pm = 36 \), \( K = 30 \) \( Pm = 18 \)

5. Conclusion

In this paper temperature control system using PID is derived mathematically and the Time response and Frequency response analysis is carried out theoretically and by simulation and different parameters are calculated.
References


