Calculation of the Exact Gain Offered by Gray over Binary Natural Code Mapping

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ABSTRACT: *The binary natural and the Gray binary reflected codes have been widely applied in many communication systems. In particular, Gray binary codes have been found to be optimum in reducing the error compared to binary natural codes. the Gray codes outperform the binary codes. In this paper the exact gain offered by Gray codes over binary codes is computed for the first time and found that the gain is dependent on the code-word length (constellation size), however at longer code-words the gain due to Gray labeling tends to remain at 3dB.*

Keywords: Binary Codes, GrayBinary Code, Natural Code, Code Mapping

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1. Introduction

The problem of selecting a binary labeling for the signal constellation for M-PSK, M-PAM, M-QAM and other types of modulation techniques has been the subject of many papers [1-2-3]. Most of communication systems use one or other varaint of modulation techniques. Universal mobile telecommunication system (UMTS) as an example of narrowband systems that apply such modulation rechniques. Similarly in wideband systems orthogonal frequency division multiplexing (OFDM) and single carrier with frequency domain equalization (SC/FDE) can be refred to.

Codewords are always referred to the signal constellation or labeling. Labeling shown in Figure 1 is more precise since the codewords in communication systems, in a signal space are a specific codewords that are labels of a set of M points in m-dimensional signal space. The transmitter selects a point from this space during each transmission interval. Each point or labeling represents a symbol of m bits. At the receiving side, an error occurs if a bit or symbol received is not what is sent. This error can be minimized if the signal constellation is properly selected and Gray binary reflected codes are found to be optimum.

2. Gray and Binary Natural Code Mapping

Gray reflected binary code, as Frank Gray first named it when he introduced it, are well known and have the property that two m-bit symbols corresponding to adjacent symbols differ only in a single bit and achieves Hamming distance of unity between any adjacent code words.

Wide range of applications apply Gray codes, in designing digital logic, it ease the design and simplify circuits. Karnaugh map method use Gray codes in labling the axes. In the field of our concern, the field of error correction in digital transmission Gray codes are extensivly used as well. As a result to Gray code nature, an error in an adjacent symbol is accompanied by one and only one bit error. Therefore, Gray codes demonstrate better performance than binary codes.That is why Gray codes are more attractive.

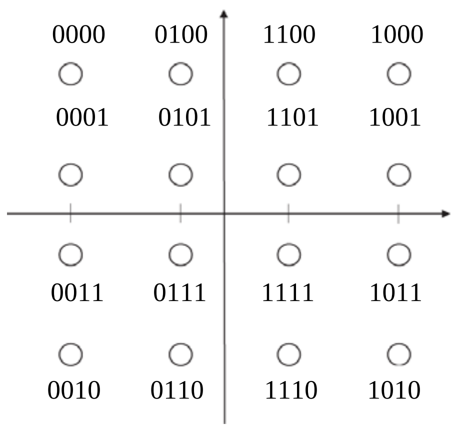

Figure 1.16-QAM signal constellation

The name, Gray reflected binary code, is derived from the fact that it may be constructed from a binary code by a reflection process.

Today, Gray codes are widely used in digital communications and many algorithms for conversion between binary code and binary reflected Gray code were developed. Agrell in [1] revised the original work of Gray and reported that Gray bit mapping for M-PSK is optimum in comparison with binary natural mapping. Transformation from binary to Gray codes or vise versa is a linear process [4].

The binary number $b_1 b_2 ... b_{(n-1)} b_n$ can be converted to its corresponding Gray binary reflected by starting from the last bit b_n and checking it if it is 1 it is then replaced by its complementary or 0, otherwise it remains unchanged. Applying the same procedures and moving backward to the first bit which usually 0.

A comparison between QPSK, 8-PSK and 16-PSK of Gray code with binary natural code bit mapping shown in table 1 reveals an interesting and useful relation, which surprisingly, has never been reported and further, finalizes the work in [1]. Let the m-tuple $\boldsymbol{b} = b_{n,0} ... b_{n,i} ... b_{n,m}$ denote the n^{th} symbol (codeword), where $m = \ln(M)$ and $n = 0, 1, ..., M-1$.

The following equation,

$$Q = \sum_{i=1}^{m} \sum_{n=1}^{M} b_{n\,i} \oplus b_{(n+1)\,i} \qquad (1)$$

gives a decimal number which represents the total number of possible transitions (change from one to zero or vice versa) for any M-PSK constellation size, as illustrated in table 1, where $\oplus$ is an XOR operation. Here, $b_{(M+1)\,i}$ is set to $b_{0\,i}$. Based on equation (1), an exhaustive search for the optimum values of Q was launched and the results are tabulated in table 2. However, it is worth mentioning that, for $m > 3$ there exist several Gray code constellations (labeling) and as m increases the number of such constellation becomes rapidly very large [5]. Nevertheless, Q_g is always optimum for any constellation (labeling). From the table, one can see $Q_b > Q_g$.

The interpretation of such a difference is, that the probability of a binary code mapping falling in an error is higher than in case of a Gray code.

Making $F = \frac{Q_b}{Q_g}$, a closer look at table 2 indicates

	Binary bits	Gray bits
1	00	00
2	01	01
3	10	11
4	11	01
	$Q_b = 6$	$Q_g = 4$

	Binary bits	Gray bits
1	000	000
2	001	001
3	010	011
4	011	010
5	100	110
6	101	111
7	110	101
8	111	100
	$Q_b = 14$	$Q_g = 8$

	Binary bits	Gray bits
1	0000	0000
2	0001	0001
3	0010	0011
4	0011	0010
5	0100	0100
6	0101	0111
7	0110	0101
8	0111	0100
9	1000	1100
10	1001	1101
11	1010	1111
12	1011	1110
13	1100	1010
14	1101	1011
15	1110	1001
16	1111	1000
	$Q_b = 30$	$Q_g = 16$

Table 1. Binary Code Bit Mapping Versus Gray Code BitMapping

$$F = \frac{2^m - 1}{2^{m-1}} \tag{2}$$

which relates the error probability of both binary natural code mapping and Gray code mapping, as

$$P_b = FP_g \tag{3}$$

where P_b and P_g are the error probabilities of binary natural and Gray codes, respectively.

As shown in Figure 2, F increases with m and tends to become constant at higher values of m according to

$$limit_{m \to \infty} \frac{2^m - 1}{2^{m-1}} = 2 \tag{4}$$

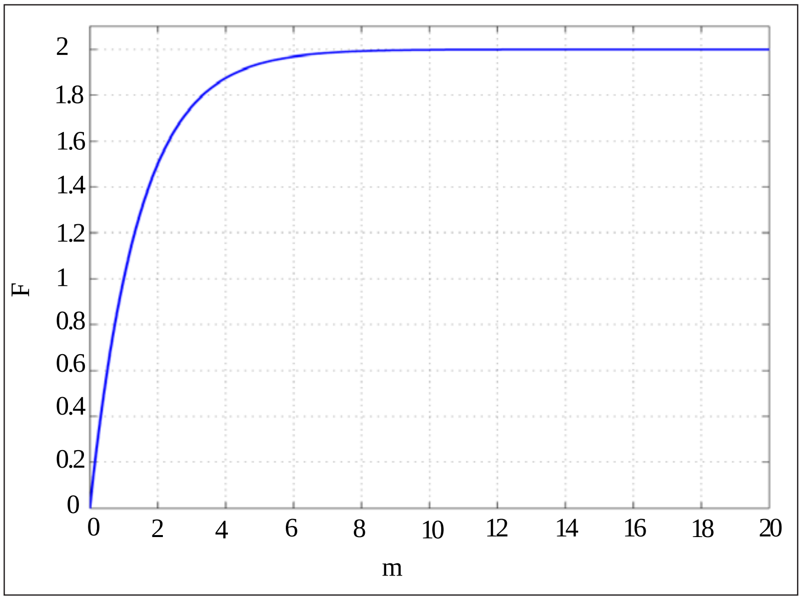

Figure 2. The marginal Figure

The importance of equation (3) stems from the fact that the evaluation of BER of one code can be extended straightforwardly to the other. Figure 3 shows the probability of error of both bit mapping techniques Gray and binary codes [6]. Further, the

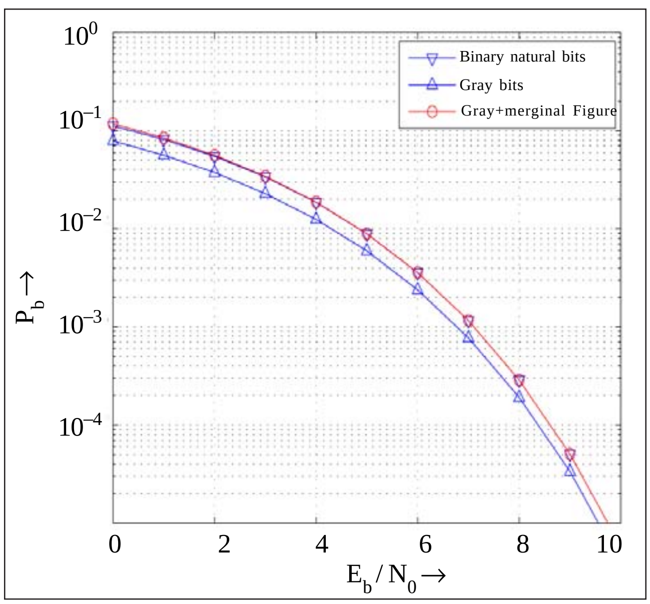

Figure 3. BER performance of both Gray and Binary Natural Code mapping for $M = 4$

m	M	Q_b	Q_g
2	4	6	4
3	8	14	8
4	16	30	16
6	64	126	64

Table 2.Optimum Number of State Transitions for Different Constelllations

probability of error of Gray code, which is obtained by the application of equation (3) (by multiplication of P_g by F) is also included. The figure shows an agreement between the simulated and the analytical result.

3. Conclusion

We can conclude from this work that in the field of digital transmission, the Gray binary reflected codes or mapping over binary naturally code offers a 3dB gain in terms of bit error for the large constellation, i.e., $m \geq 8$. However for $m = 2$, 4 and 6, the improvements are precisely 1.7, 2.5 and 2.9 dB, respectively. Further, for large constellations this gain is found to be independent on the size of the constellation.

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