Image Zooming Model Based on Fractional-Order

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ABSTRACT: *This paper proposes a new image zooming model based on the fractional-order partial differential equation, which adopts the idea of total variation. The simulation results show that this new model can keep the characteristics of image edge better than the fourth order partial differential equation model since it can retain more texture details. The numerical results shows that the proposed model is effective and practical on image zooming.*

Keywords: Fractional-order, Total variation, Image zooming.

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1. Introduction

In recent years, many experts are interested in studying image model based on PDE, which is widely used in image enhancement, image reconstruction, image segmentation, image zooming [1-4] and other fields. Image restoration is an important factor to measure the image visual effect. We often need to obtain image processing (such as image scaling or rotation) which is be applied in the special field (such as medical, communications, aerospace etc.) and some image processing software, aimed to increase the image resolution while preserving the image visual effect of high quality.

Images often need to be zoomed in or out reproduced to higher resolution from lower resolution. One common way for image zooming is gray-scale interpolation. If we only process the pixels for image, the result is that the image appearing staircase and blocky effect, and the zooming effect is worse for the image with noise; but the image zooming method based on PDE can effectively suppress the influence of blocky effect and noise[5,6]. The algorithm of image zooming based on PDE can be divided into two categories-direct method and indirect method[7].This paper uses the direct method, first we take the traditional interpolation result as the initial value of the enlarged image, then we process the image with application of PDE .Based on the Rudin, Osher and Fatemi(ROF) model appeared in [8] and the model proposed by Lysaker, lundervold and Tai(LLT) in [9], the author have proposed a fourth-order PDE image zooming model [10]as follow :

$$E(u) = \int_{\Omega} (u_{xx}^2 + u^2_{xy} + u^2_{yx} + u^2_{yy})^{\frac{1}{2}} \, dx \, dy + \frac{\lambda}{2} \int_{\Omega} (u - u_0)^2 \cdot \chi_{\Omega_1} (u - u_0) \, dx \, dy \quad (1)$$

where χ_{Ω_1} is the characteristic function of Ω_1, $(x, y) \in \Omega_1$ is a pixel of the original image. The simulation results show that this algorithm not only can eliminate jagged edge and texture regions, but also eliminate the sharpness and blocky effect due to image zooming. But the enlarged image becomes blurring, and at the same time visual defects such as clarity is not enough.

In order to solve the image blur and detail characteristics loss problems, this paper proposes an image zooming algorithm based on fractional-order PDE[11-13]. Fractional-order differential equation has significantly improved the image with high-frequency components, enhanced the image with frequency components, nonlinear preserving image characteristics of the low-frequency components, which retain more image gray-scale texture details have not changed much in the smooth region, making the enlarged image clarity, better keeping the original image edge features.

Section 2 of the paper gives the related theory of fractional-order derivative. We formally introduce our algorithm in Section 3. Section 4 is devoted to the implementation details and numerical examples, followed by some conclusions giving in Section 5.

2. The related theory

2.1 Definition of the fractional-order derivative

The fractional-order derivative is a generalization of integer order derivative. Although the fractional-order derivative has been widely used in physics, chemistry, biology, fluid and other fields, but its definition is not uniform. There are three kinds of classical definitions, including Riemann-Liouville(R-L), Capotu and Grumwald-Letnikov(G-L).

Here, we give the definition of G-L fractional-order derivative:

$$D^p f(x) = \lim_{h \to 0} \frac{\sum_{k \geq 0}^{n} (-1)^k \binom{p}{k} f(x - kh)}{h^p} \quad (2)$$

where p is the order of fractional-order derivative,

$$\binom{p}{k} = \frac{\Gamma(p+1)}{\Gamma(k+1)\,\Gamma(p-k+1)}$$

where $\Gamma(\kappa) = \int_0^{+\infty} x^{k-1} e^{-x} dx$ is Gamma function, $\Gamma(k + 1) = k\,\Gamma(k)$, if k is an integer, then $\Gamma(k + 1) = k!$. Obviously, when p=1, the above formula is the usual sense of the first derivative.

For fixed p, $\lim_{k \to 0} \binom{p}{k} = 0$.

when $h = 1$, we can use limited items (such as pre- K items) of fractional difference to approximate the G-L fractional-order derivative, i.e.:

$$D^p f(x) \approx \sum_{k=0}^{k-1} (-1)^k \binom{p}{k} f(x - k), \quad (3)$$

2.2 Difference scheme of fractional-order derivative

The above definition can be extended to two-dimensional case, and the fractional-order partial derivative is defined in the following:

$$\frac{\partial^p u}{\partial x^p} = \lim_{h \to 0} \frac{\sum_{k \geq 0}^{n} (-1)^k \binom{p}{k} u(x - kh, y)}{h^p} \quad (4)$$

$$\frac{\partial^p u}{\partial y^p} = \lim_{h \to 0} \frac{\sum_{k \geq 0}^{n} (-1)^k \binom{p}{k} u(x, y - kh)}{h^p}, \quad (5)$$

$$\frac{\partial^p u(x,y)}{\partial x^p} \approx u(x,y) + (-p)\, u(x-1,y) + \frac{-p(-p+1)}{2} \, . \, u(x-2,y) + \ldots + \frac{\Gamma(-p+1)}{n!\, \Gamma(-p+n+1)} u(x-n,y), \quad (6)$$

$$\frac{\partial^p u(x,y)}{\partial y^p} \approx u(x,y) + (-p)\, u(x,y-1) + \frac{-p(-p+1)}{2} \, . \, u(x,y-2) + \ldots + \frac{\Gamma(-p+1)}{n!\, \Gamma(-p+n+1)} u(x,y-n). \quad (7)$$

3. Image zooming based on fractional-order derivative

We first process images by using linear interpolation, enlarging the image to the desired ratio, and then apply the technique of fractional-order PDE for the enlarged image, finally, achieve the purpose of the image zooming.

First, we give the energy functional of the model:

$$E(u) = \int_\Omega \left| D^p u \right| dx\, dy + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 \cdot \chi_{\Omega 1} \, (u - u_0) \, dx\, dy \quad (8)$$

where the first item of the energy functional is the regularization term; the second term is the fidelity term, $\chi_{\Omega 1}$ is the characteristic function of Ω_1, $(x, y) \in \Omega_1$ is a pixel of the original image. $\lambda > 0$ is a constant. It is through repeated experiments and artificially selected parameters to balance these two effects.

Using the variation method, we can obtain the Euler-Lagrange of (8) as following:

$$-D^p . \left(\frac{D^p u}{\left| D^p u \right|} \right) + \lambda (u - u_0) \cdot \chi_{\Omega 1} (u - u_0) = 0, \quad (9)$$

where $D^p = \left(\frac{\partial^p}{\partial x^p}, \frac{\partial^p}{\partial y^p} \right)$.

According to the gradient descent method, we can obtain the fractional-order PDE with the following boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} = D^p \left(\frac{D^p u}{\left| D^p u \right|} \right) + \lambda (u - u_0) \cdot \chi_{\Omega_1} (u - u_0) \\ u(0, x, y) = u_0(x, y) \\ \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0 \end{cases} \quad (10)$$

4. Implementation details and numerical results

4.1 Implementation details

To discretize the equation (10), we use finite differences. Let Δt be the time step. We denote by u^n the approximations for $u(x, y, n\Delta t)$. Where x and y are the grid points. The approximations we have used in our scheme is outlined in Table 1.

$D_x^p u_{ij}^n$	$u_{ij}^n + (-p)u_{i-1,j}^n + \frac{-p(-p+1)}{2}u_{i-2,j}^n$
$D_y^p u_{ij}^n$	$u_{ij}^n + (-p)u_{i,j-1}^n + \frac{-p(-p+1)}{2}u_{i,j-2}^n$
$\lvert D^p\left(u_{i.j}^n\right)\rvert$	$\sqrt{\left(D_x^p\left(u_{i.j}^n\right)\right)^2 + \left(D_y^p\left(u_{i.j}^n\right)\right)^2 + \varepsilon}$

Table 1. discretization used in the implementations

To simplify the notations, we will omit the subscripts i, j and use u^n to denote u_{ij}^n. The details of the algorithm we have used are given in the following:

Algorithm: We assume initial image matrix T_0 is $m \times n$.

(1) By sampling to reduce k (k is a positive integer) times in order to get T', assigning T' to T, and then using linear interpolation for T (this paper adopts three cubic interpolation), finally, we get the matrix u_0 which is to be magnified for k times;

We find u (the purpose of the image zooming) by (10) and update

$$u^{n+1} = u^n + \Delta t\left[D_x^p\left(\frac{D_x^p u^n}{\left|D_x^p u^n\right|}\right) + D_y^p\left(\frac{D_y^p u^n}{\left|D_y^p u^n\right|}\right)\right] + \Delta t\lambda\left(u^n - u_0\right)\cdot\chi_{\Omega_1}\left(u^n - u_0\right), \quad (11)$$

where u not only keeps the image of the original features, but also maintains the continuity between different gray-scale.

4.2 Numerical Results

In the experiment, we choose $\lambda = 0.5$, $\Delta t = 0.05$, $p = 1.7$ in formula (11), and set the iteration is 40 times. In order to show numerical evidence of our algorithm, the Lena image with the size 256×256 is reduced to 2 times by sampling, and then is zoomed.

Figure 1 shows that image zooming algorithms based on fractional-order partial differential equation has better results compared with fourth-order PDE model and maintain the original features of the image when the image is enlarged.

To further illustrate the difference of image zooming algorithms between fractional-order partial differential equation and fourth-order integer partial differential equation, we take the comparison of the gray curve distribution of the 150th of original image, the image zoomed based on fractional-order and the image zoomed based on fourth-order PDE as follow:

Figure 2 shows the difference of image zooming algorithm between integer and fractional-order: When using the amplification of fourth-order PDE model, although the overall effect is to get better, but due to excessive smooth, so that the edges blur and some details are not clear enough. The interpolation model of fractional partial differential equations truly reflects the original features of the image, maintain the sharpness of edges and texture characteristics.

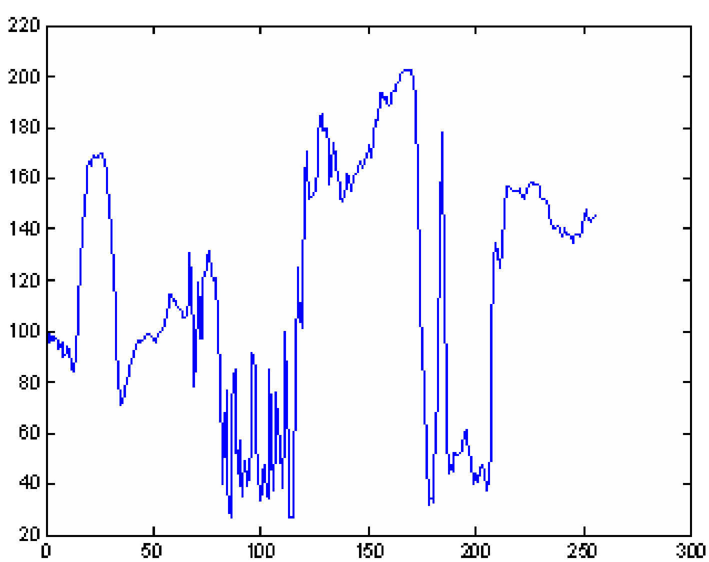
(a) Original

(b) Down sample

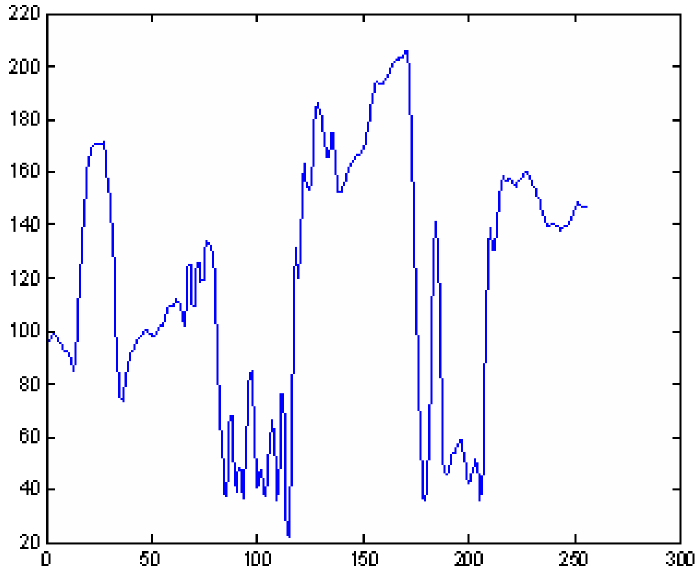
(c) image zooming
by fractional-order PDE

(d) image zooming by fourth-order PDE

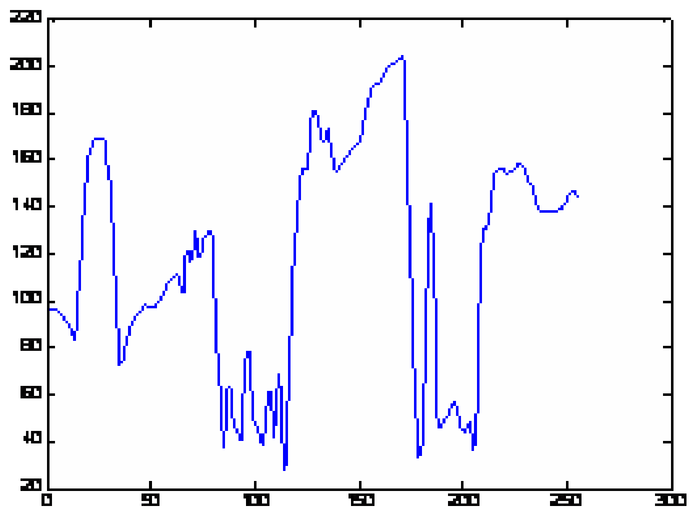
Figure 1. Results of image zooming



(a) Original image



(b) The image zoomed based on fractional-order



(c) The image zoomed based on fourth-order PDE

5. Conclusions

This paper describes a method for image zooming. The proposed method based on fractional-order PDE. The algorithm can maintain edge features and details of the image better. Meanwhile the computation time of the fractional-order PDE is less than that of the fourth-order PDEs. So the proposed algorithm is feasible and effective, which is confirmed by numerical experiments. However, this paper only use a fixed number as the order of the fractional differential, the future study will be carry out about how to determine a better differential order to obtain a better effect.

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