Identification of a Fractional Order Model by a Least Squares Technique: H_n Model

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ABSTRACT: *We report on fractional modeling and identification of non-integer systems by least squares method (LS). A new approach to identify fractional differentials equations (FDE) is proposed. Such a technique presents a linear model to estimate system parameters, as well as non-integer orders from temporal data (Hn model). The identified parameters provide original solution to the Output Error method initialization (OE), and demonstrate validity and effectiveness of the proposed approach.*

Keywords: Non-integer system, Fractional Integration, Modeling, Identification, Output Error Method, Least Squares Method

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1. Introduction

Fractional calculus has attracted a great deal of attention of researchers and mathematicians and there has been a rapid grow in the number of applications where fractional calculus has been used. Indeed, this technique has been applied to physics and engineering science problems and has been widely studied in different fields of science such as the electrotechnical, automation, image processing, and chemicals whether for modeling problems of identification or control.

The high quality of the description of physical phenomena, provided by the fractional modeling compared to the entire modeling for asynchronous machines is the main motivation of this work.

In this context, we present a new identification approach that extend the method of least squares to non-integer order systems after determining a linear model with respect to parameters, based on repeated fractional integration. The obtained parameters constitute an original solution to the initialization of Output Error Method problem. A comparative study with arbitrary initialization was also proposed to highlight the interest of the present study.

In part 2, 3 and 4, we refer to more details and results about the fractional integration operator, FDE simulation. Then the least squares method (LS) is used to identify the FDE after model linearization (part 5, 6 and 7). Results of LS method are used to initialize the OE technique in part 8 and demonstrate the improvement of this technique compared to the arbitrary initialization parameters. Finally, we achieve by the numerical simulation to justify validity and effectiveness and validity of the proposed method.

2. Fractional Integration

The n^{th} fractional order Riemann–Liouville integral (n real positive) of a function $f(t)$ is defined by the relation (1):

$$I_n(f(t)) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} \tag{1}$$

Where $\Gamma(n)$ is the gamma function.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{2}$$

$I_n(f(t))$ is the convolution of the function $f(t)$ with the impulse response:

$$h_n(t) = \frac{t^{n-1}}{\Gamma(n)} \tag{3}$$

Of the fractional integration operator whose Laplace transform is:

$$I_n(s) = L\{h_n(t)\} = \frac{1}{s^n} \tag{4}$$

Notice that in the integer order case (n = 1), the integral is characterized by $h_1(t) = H(t)$ (unit step function or Heaviside function) and

$$I_1(s) = L\{h_1(t)\} = \frac{1}{s} \tag{5}$$

Fractional differentiation is the dual operation of fractional integration.

$$x(t) = I_n(v(t)) \quad \text{or} \quad X(s) = \frac{1}{s^n} V(s) \tag{6}$$

Reciprocally, $v(t)$ is the nth order fractional derivative of $x(t)$ defined as:

$$v(t) = D_n(x(t)) \quad \text{or} \quad V(s) = s^n X(s) \tag{7}$$

where $D_n(s) = s^n$ represents the Laplace transform of the fractional differentiation operator (for initial conditions equal to zero).

This fractional derivative definition is based on the operator $I_n(s)$, without analytical formulation of $D_n(x(t))$: it is the implicit definition of the fractional derivative.

3. Synthesis of the Fractional Integration Operator

Simulation and identification of fractional differential equations (FDE) model is fundamentally based on the fractional integration operator I_n. At the end to synthesize this key operator a comparative study is presented between two different approaches in the frequency and temporal domain.

However, the realization of $I_n(s)$, either in analog or numerical form, is not a simple task, as in the integer order case. The reader will refer to [3 and 4] for a more detailed presentation.

3.1 Diffusive representation of the fractional integrator

The impulse response $h(t)$ can be expressed by:

$$h_n(t) = \int_0^\infty \mu(w) e^{-wt} dw \tag{8}$$

Where $\mu(w)$ is the diffusive representation of $I_n(s)$ (or function of frequency representation).

Laplace transform of h (t) is given by:

$$H(s) = \frac{1}{s} = L\{h(t)\}, 0 < n < 1 \quad (9)$$

Equation (8) is equivalent to:

$$h(t) \cong \sum_{k=1}^{K} c_k e^{-w_k t} \quad (10)$$

Where:

$$c_k = \mu(w_k)\,\Delta_{wk} \quad (11)$$

Many approaches can be used to define the conventional fractional operator. In practice, the model which simplifies the numerical simulation of the integrator is used. The fractional order integrator is an infinite dimensional system. The state space theoretical model of the integrator can be expressed as [1, 5, and 6]:

$$\begin{cases} \dfrac{\partial x(t,w)}{\partial t} = -wx(w,t) + u(t) & (12) \\ y(t) = \int_0^{\infty} \mu(w)\,x(w,t)\,dw & (13) \\ \mu(w) = \dfrac{sin(n\pi)}{\pi} w^{-n}, 0 < n < 1 & (14) \end{cases}$$

Where $u(t)$: input, $y(t)$: output, $x(w, t)$: continuously distributed state of $I_n(s)$.

3.2 Frequency discretized distributed model

In practice, in order to obtain a finite dimensional approximation, one proceed to a frequency discretization of the distribution function $\mu(w)$ as reported in Figure 1. Such solution is related to the impossibility to use directly the continuous frequency weighted model given by relations 12, 13 and 14.

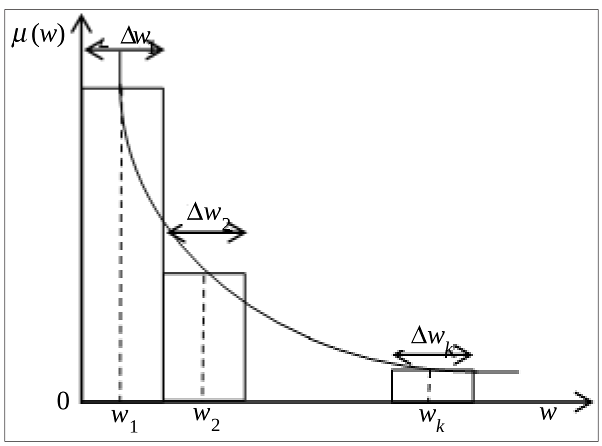

Figure 1. Frequency discretization of $\mu(w)$

By replacing $x(w)$ by a multiple step function (with K steps) $x(w_k, t)$ in Equation 12 and 13, one obtain after discretization [1, 2 and 7]:

$$\begin{cases} \dfrac{dx_k}{dt} = -w_k x_k(t) + u(t) \\ y(t) = \sum_{k=1}^{K} \mu(w_k)\,x_k(t)\,\Delta_{wk} \\ \quad\;\; = \sum_{k=1}^{K} c_k x_k(t) \end{cases} \quad (15)$$

For an elementary step, its height is $\mu(w_k)$, and its with is Δw_k. Let c_k be the weight of the k^{th} element:

$$c_k = \mu(w_k)\,\Delta_{wk} \quad (16)$$

Or equivalently to:

$$\begin{cases} \underline{\dot{X}}(t) = A\underline{X}(t) + \underline{B}u(t) \\ y(t) = \underline{C}^T \underline{X}(t) \end{cases} \quad (17)$$

Where:

$$\underline{X}(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_K \end{bmatrix}; A = \begin{bmatrix} -w_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & -w_K \end{bmatrix}; \underline{B}^T = [1 \quad 1 \ \ldots \ 1]; \underline{C}^T = [c_1 \ \ldots \ c_K]$$

This approach is characterized by the simplicity of its state representation. w_K represents modes of the state representation. These modes are ranging from w_1 (<< 1) up to w_K (>> 1).

The Infinite State representation can be schematized by the graph of Figure 2, where the frequencies ω_k, ranging from $\omega_0 = 0$ to ω_K, are the modes of the fractional integrator which act in parallel. Notice that the integer order integrator is characterized by only one model located in $\omega_0 = 0$.

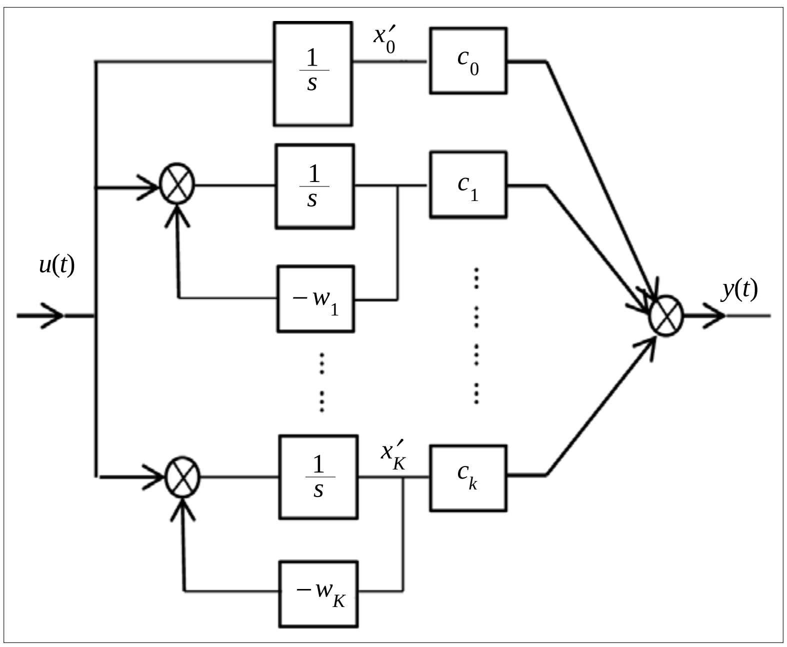

Figure 2. The modal representation (Infinite State representation) of fractional integrator

4. FDE Simulation

Before resolving identification problem of fractional systems, we should firstly take into account both modeling and simulation of FDE. Fractional integration (order *n*) is based on representative state system with *n*-ordinary differential equations. This approach will be generalized for FDE based on fractional integrator, considered as a key for fractional system simulation [8 and 9].

For this purpose, let consider following elementary FDE (18) corresponding to non-integer system given by Equation 19:

$$\frac{d^n y(t)}{dt^n} + a_0 y(t) = b_0 u(t) \quad (18)$$

$$\frac{Y(s)}{U(s)} = \frac{b_0}{a_0 + s^n} \quad \text{Pour } 0 < n < 1 \quad (19)$$

In that case the corresponding state representation is given by:

$$\begin{cases} \frac{d^n x(t)}{dt^n} + a_0 x(t) = u(t) \\ y(t) = b_0 x(t) \end{cases} \qquad (20)$$

Based on fractional integrator $I_n(s)$, simulation with block representation scheme (see Figure 3) are performed [10, 11 and12].

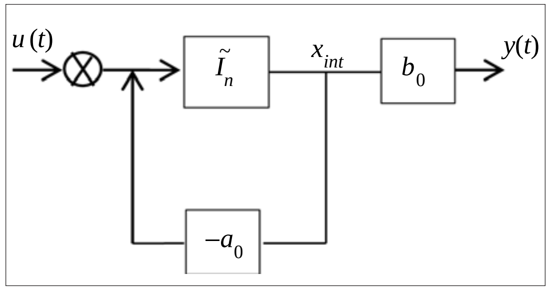

Figure 2. Block representation scheme of non-integer system ($0 < n < 1$)

5. Least Squares Method and FDE

The least squares method (*LS*) applied to the identification consider the linear model with respect to parameters (*LP* Model) given by equation (21):

$$\hat{y}_k(t) = \varphi_k^T \hat{\theta} \qquad (21)$$

Where: φ^T, θ and $\hat{y}_k$ are regression vector, estimated parameters and estimated measure respectively.

By minimizing the quadratic criterion function J, estimation of θ_{MC} can be obtained (Equation 23) [13].

$$J = \sum_1^K (y_k^* - \hat{y}_k)^2 \qquad (22)$$

$$\theta_{MC} = (\sum_1^K \varphi_k \varphi_k^T)^{-1} \sum_1^K \varphi_k y_k^* \qquad (23)$$

Where y_k^* represents the measure (k varies from 1 to K).

Alternately, the least squares method (LS) handles with linear models, which is not the case for fractional differential equations. To overcome such restriction, the FDE was linearized in order to estimate parameters model with mentioned method (LS).

6. Determination of LP Model with FDE

The idea proposed in this section is to determine a linear model with respect to parameters, based on repeated integration. Indeed, the fractional system for the Hn model can be expressed by:

$$D_n(y) + a_0 y = b_0 u \qquad (24)$$

Then by integration of the differential equation one obtains:

$$I_n(D_n(y)) + a_0 I_n(y) = b_0 I_n(u) \qquad (25)$$

By considering Laplace transform of $D_n(y)$ expressed by Equation 26, we obtain $L(I_n D_n(y))$ (Equation 27).

$$L(D_n(y)) = s^n Y(s) - s^n \int_0^\infty \mu_n(\omega) \frac{z(\omega,0)}{s+\omega} d\omega \qquad (26)$$

$$L(I_n D_n(y)) = Y(s) - \int_0^\infty \mu_n(\omega) \frac{z(\omega,0)}{s+\omega} d\omega \qquad (27)$$

Where $z(\omega, 0)$ is the initial condition of FDE referred as $y(0)$ for ordinary differential equation. Then, the inverse Laplace transform is given by:

$$L^{-1}(I_n D_n(y)) = y(t) - \int_0^{\infty} \mu_n(\omega)\, z(\omega, 0)\, e^{-\omega t} d\omega \quad (28)$$

Where:

$$\int_0^{\infty} \mu_n(\omega)\, z(\omega, 0)\, e^{-\omega t} d\omega \quad (29)$$

is the fractional equivalent of $y(0)$.

It is supposed that the system is at rest at all frequencies, which can approximate $z(\omega, 0) = 0$, whatever the value of the pulsation ω.

This allows us to write:

$$I_n(D_n(y)) = y(t) \quad (30)$$

Consequently:

$$y(t) + a_0 I_n(y) = b_0 I_n(u) \quad (31)$$

The corresponding matrix transformation is given by:

$$y(t) = [-I_n(y) \quad I_n(u)] \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \varphi^T \theta \quad (32)$$

Equation 32 shows that we get a linear model with respect to parameters, based on repeated integration. These results allow applying the LS method and estimating parameters model.

7. Numerical Simulation

The estimated parameters $\theta(n)$ corresponding to each imposed value n is determined by *LS* method. Then the quadratic criterion relative J is deduced. The θ_{MC} values containing a_{MC} and b_{MC} corresponding to the minimum value of J are the estimated parameters. The sampling period of n is 0.1 and it can be refined further. Measurement data for identification is considered firstly noiseless and then as a noisy signal generated by superposition of the Gaussian white noise. Simulations are performed with Matlab software.

7.1 Noiseless identification results of fractional model (H_n)

In this part, results of noiseless identification results of Hn model are investigated by LS method. Simulated fractional model are performed by the following parameters $a_0 = 2$, $b_0 = 1$, $n = 0.6$. These selected parameters must ensure system stability.

The fractional integrator is simulated for a number Nb_sim cells, valid in a frequency band [wb_sim, wh_sim], and a sampling period Te_sim where Nb_sim = 15, wb_sim = 10^{-3} rd/s and wh_sim = 10^{3} rd/s.

The fractional integrator of regression calculation is simulated for a number of cells Nb_reg, valid in a frequency band [wb_reg, wh_reg] and a sampling period Te_reg.

The system is excited by a pseudo-random binary signal (PRBS) (amplitude equal to 3). The sampling period is set to Te_sim = 10^{-3}s.

Measured output of the system and those estimated by *LS* method, excited by the same input are shown in Figure (3). The insert in figure 3 represent the area of zoom.

Figures (4) and (5) respectively represent the variation of the mean squares error as a function of the order n of the system and the variation in the error between the exact response and the estimated models. The insert in figure 4 represent the area of zoom.

Results of the identification of H_n model are presented in Table (1) for Te_reg = Te_sim and Te_reg = 5Te _sim.

The parameters *a* and *b* estimated for the two tests (for Te_reg = Te_sim and Te_reg = 5Te_sim) are relatively close and accurate,

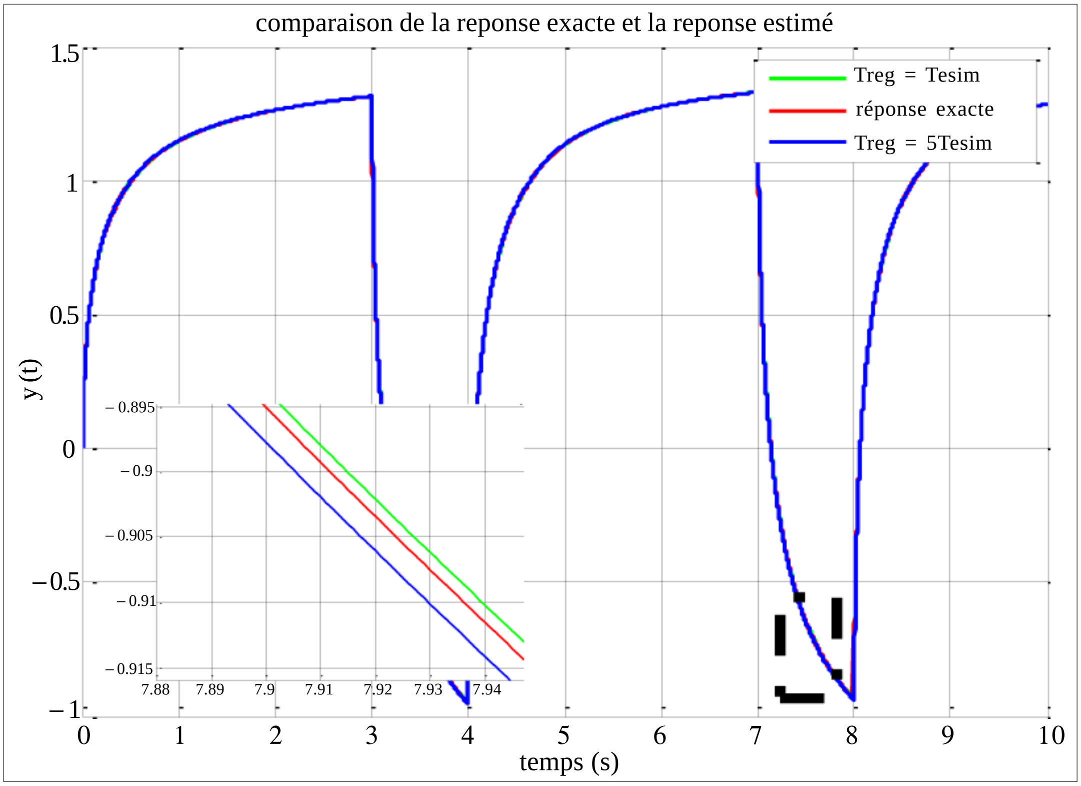

Figure 3. Measured output of the system and those estimated
by *LS* method Te_reg = Te_sim and Te _reg = 5 Te_sim

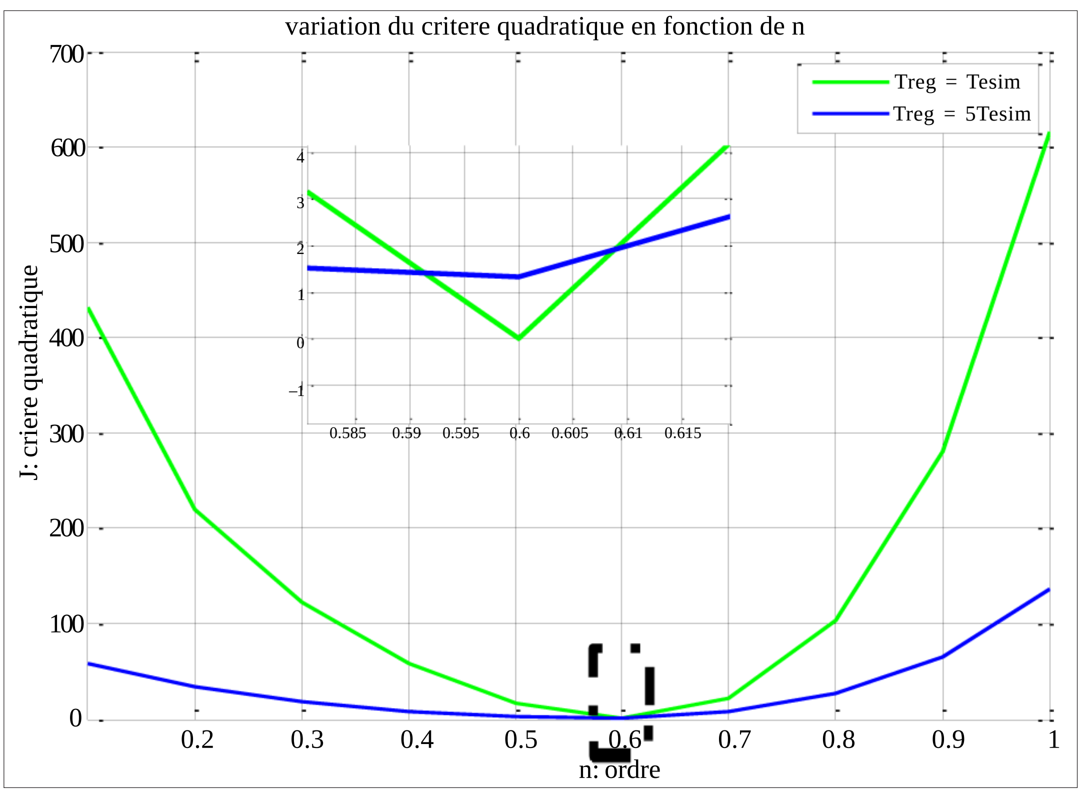

Figure 4. Variation of the mean squares error, Te_reg = Te_sim and Te _reg = 5 Te_sim

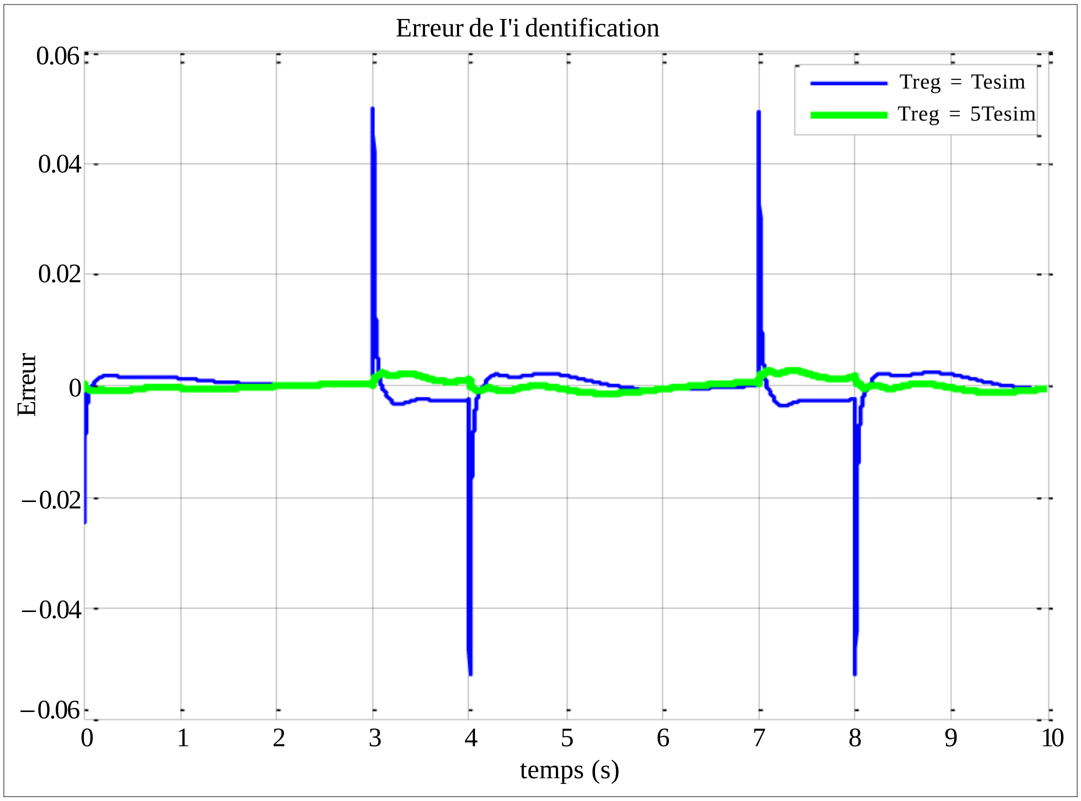

Figure 5. Error variation between the exact response and the estimated models: Te _reg = Te_sim et Te_reg = 5 Te_sim

	Te_reg = Te_sim			Te_reg = 5Te_sim		
Parameters	***a*0**	***b*0**	***n***	***a*0**	***b*0**	***n***
Simulation	2	1	0.6	2	1	0.6
Estimation	1.994	0.997	0.6	2.008	1.004	0.6
Quadratic error	0.01			1.3165		

Table 1. Noiseless Identification of H_n Model Estimated by the Least Squares Method for Te_Reg = Te_Sim And Te_Reg = 5te_Sim

it is similar to the order *n*. In addition, the temporal response of both simulated and estimated model reveals a great similarity.

Commitantly, we observe a small periodic error in the range of 0.05 and – 0.05, due to modes truncation. On the other hand, the quadratic error increases gradually as one move away from the exact order and rapidly decreases when it approaches.

7.2 Noisy identification results of fractional model (H_n)

In this section we introduce a white Gaussian noise to the measurement data and we will take over the identification. The identification of Hn model is carried out of 100 noisy realizations. The output of the exact model and the estimated model are shown in Figure (6). Results show good agreement between the two models.

Table (2 and 3) shows the average of each identified parameter and the standard deviation (distribution of Monte Carlo) for Te_reg = Te_sim and Te_reg = 5Te_sim respectively.

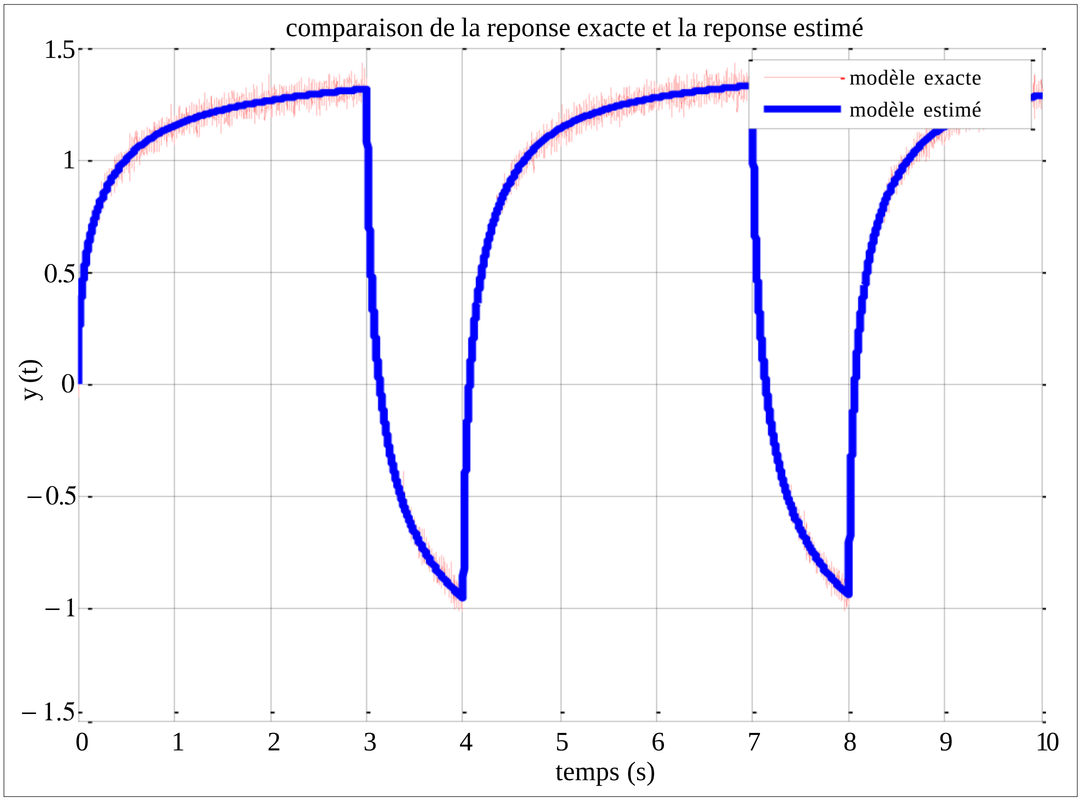

Figure 6. Output of the exact model and the estimated model: Te _reg = 5Te_sim and SB = 30

	SB = 30			SB = 20		
Parameters	a_0	b_0	n	a_0	b_0	n
Simulation	2	1	0.6	2	1	0.6
Average	1.9940	0.9975	0.6	1.9937	0.9973	0.6
Variance	0.0029	0.0010	$8.926^e - 16$	0.0085	0.0032	$8.926^e - 16$

Table 2. Noisy Identification of H_n Model Estimated by the Least Squares Method for Te_Reg = Te_Sim

	SB = 30			SB = 20		
Parameters	a_0	b_0	n	a_0	b_0	n
Simulation	2	1	0.6	2	1	0.6
Average	2.0090	1.0040	0.6	2.0037	1.0021	0.5990
Variance	0.0063	0.0025	$8.926^e - 16$	0.0477	0.0164	0.01

Table 3. Noisy Identification of H_n Model Estimated by the Least Squares Method for Te_Reg = 5te_Sim

8. Iinitialization of output Error Method

As a first step, we will proceed to the identification by the output error method of H_n model associated to the Levenberg-

Marquardt optimization procedure which considering a direct initialization (arbitrary initialization) [6]. The measurement data used in the identification is assumed noiseless. Thereafter, it will take over the identification using the values provided by *LS* method for parameter initialization.

The simulation evidences that the direct initialization method of the model is very delicate, and highlights the utility of the *LS* initialization. Before addressing the identification procedure by the output error method.

The H_n model (Equation 19) is simulated with the parameters found in paragraph (7), and we verified model stability. The identification by output error method is carried out at the beginning, with arbitrary initial values. Thereafter, we will extract initial parameters range of ai, bi and ni that permit estimated values convergence to the correct one. Accordingly, we give a simulation example.

The initial value (a_i) is fixed among the three parameters to be estimated, the second one (b_i) is varied, and the range of last parameter (n_i) is determined to ensure algorithm convergence of output error toward the correct value. Tables 4 summarize simulation results.

Parameters	Initial values of a_i, b_i and n_i		
a_i	5	5	5
b_i	0.2	1	4
n_i	[0.17 ; 1.2]	[0.13 ; 1.1]	[0.13 ; 0.89]

Table 4. Range of Initial Parameter N_i

As it can be seen that a slight variation in the value of one parameter modify the range of the other one. Indeed, we cannot identify initialization parameters range separately, but we are talking about a domain combination.

The problem here is simple, but it is accentuated when the system order increases. In addition, the range which ensures the convergence criterion becomes very close when it comes to a three or four fractional integrators. This prevents any manual initialization especially when ignores all exact orders.

For this reason, the H_n model is identified using the initialization parameters provided by the *LS* method. The output $y(t)$ is considered firstly. Identification results are shown in Figure 7.

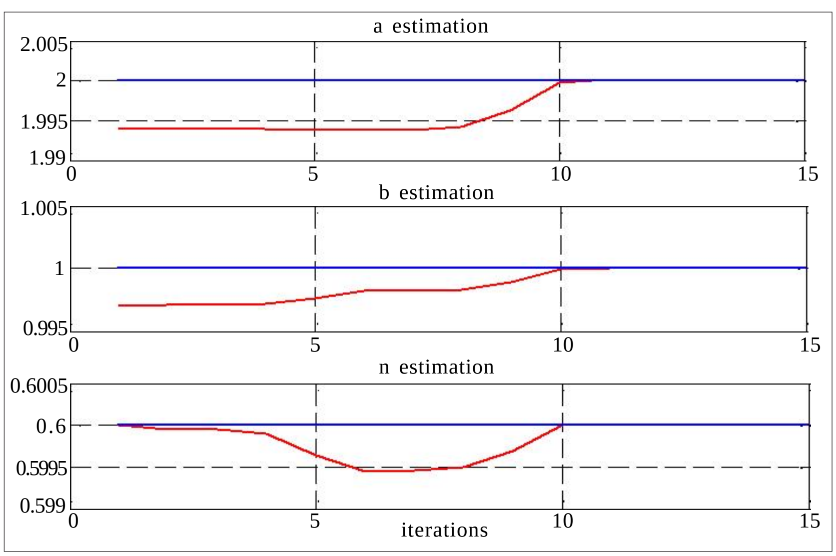

Figure 7. Initialization by the least squares method

Initialization by the least squares method avoids the arbitrary initialization that does not always offers good results. It simplifies very well process initialization, which is considered as difficult problem especially when the order of the system increases.

9. Conclusion

To summarize, we report on non-integer integration operator to modeling and simulation of fractional systems. Based on repeated fractional integration, a new identification approach that extends least squares method to non-integer order systems is proposed. This allows identifying, not only the coefficients of the system, but also non-integer orders from Hn temporal data. On the other hand, estimated parameters constitute an original solution to the initialization of Output Error Method problem. A comparative study with arbitrary initialization is also performed in order to underline the interest of the present study.

Finally, simulation results show good agreements between the exact model output and the estimated model and demonstrate validity and effectiveness of the proposed approach. Future work will address parameters identification of an asynchronous machine.

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