# Generate Synthetic ECG Signal Normal and Pathological using Mathematical Model

Fatima Guendouzi, Mokhtar Attari
Laboratory of Instrumentation (LINS)
USTHB Algiers, Algeria
{guendouzi.f, attari.mo}@gmail.com

***ABSTRACT:*** *In telemedicine, the transmission of the cardiac signal or for the diagnosis of an automatic Holter, it is important to model the heartbeat. Our aim in this work is the modeling of the ECG data by polynomial transform. We have developed an algorithm that allows the modeling of the ECG signal with the Chebyshev polynomial. The principle of ECG data modeling is presented and the relevant examples for the capability of the method are provided. The modeling algorithm is evaluated using the database of MIT-BIH. The first results obtained exhibit the faithfully reproduction abnormalities included in the ECG signal.*

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## 1. Introduction

ECG signal represents electrical changes on the skin that are caused by the heart muscles, and is usually measured by the electrodes placed on body surface. In signal processing, modeling is used mainly to produce a classification of the observed signals using the model parameters.

Due to high importance of accurate modeling several ECG modeling approaches applicable for different purposes like heartbeat synthesis, analysis, compression, and filtering were introduced. Modeling with Hermite functions is widely used in the literature for the classification of QRS complexes [1, 2] and [3]. There are also Markov models to facilitate the automatic interpretation of the ECG [4]. Polynomials of maximum degree 3, including splines functions have been proposed for ECG interpolation in [5] and [6]. In [7] Nygaard et al studied the representation of ECG signals using second degree quadratic polynomials. High degree polynomial approximations of a signal is similar to spectral methods since the signal is decomposed into a set of orthogonal polynomials basic functions, the same way to Fourier Transform and Wavelets Transform. Although Chebyshev polynomials are widely used in mathematical interpolation and in spectral methods for solving differential equations systems, propositions for ECG modeling through Chebyshev polynomials are hardly encountered in the literature [8]. In this paper we have developed an algorithm that allows the modeling of the ECG signal, normal and pathological, with the Chebyshev polynomial.

The rest of this article is organized as follows: in the next section, we give a few reminders on the theory of the orthogonal polynomials a brief introduction to Chebyshev polynomials. The implementation in order to achieve ECG modeling is described in section 4. The results obtained are presented and discussed in section 5. At the last section, conclusions are provided.

## 2. Theory of Orthogonal Polynomials

Orthogonal polynomials are usually defined by reference to the notion of integral with a weight function $\omega(x)$, positive on the interval of integration $[a, b]$. These are families of polynomials $\{y_n\}_{n\in N}$ such that:

$$\int_a^b y_i(x)\, y_j(x)\, \omega(x)\, dx = K_{ij}\delta_{ij} \tag{1}$$

With $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$K_{ij}$ is a constant which depends on $i$, $j$ and standardization adopted for the family of polynomials. The interval $[a, b]$ can be infinite. That $\omega(x)$ is a positive function on $[a, b]$ ensures the existence and uniqueness of the sequence of polynomials $\{y_n\}_{n\in N}$ from a normalization has been chosen [9].

There are three major families of classical orthogonal polynomials:

### 2.1 The Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined on the interval $[-1, 1]$ with the weight function

$$\omega(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{2}$$

Two particular classes of Jacobi polynomials are respectively the Legendre polynomials $P_n(x) = P_n^{(0,0)}(x)$ and Chebyshev polynomials of the first kind $T_n(x) = (n!\Gamma(1/2) / \Gamma(n + 1/2))\, P_n^{(-1/2,\,-1/2)}(x)$ where $\Gamma(x)$ is the gamma function defined by:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \tag{3}$$

Weight functions for the Legendre polynomials and Chebyshev polynomials are deduced from (2) and (3) $\omega(x) = 1$ and $\omega(x) = 1/\sqrt{(1-x)^2}$

### 2.2 The Laguerre Polynomials

The Laguerre polynomials $L_n^{\infty}(x)$ defined on the interval $[0, +\infty]$ and their weight function is:

$$\omega(x) = x^{\alpha} e^{-x} \tag{4}$$

### 2.3 Hermite Polynomials

Hermite polynomials $H_n(x)$ defined on $[-\infty, +\infty]$ with the weight function

$$\omega(x) = e^{-x^2} \tag{5}$$

Hermite functions are deduced from the Hermite polynomials by the expression (6)

$$h_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} H_n(x)\; e^{-1/2x^2} \tag{6}$$

Hermite functions form a complete orthogonal family.

### 2.4 Series of Orthogonal Polynomials

Development in Fourier series can be generalized and applied in the theory of orthogonal polynomials according to a theorem stated and proved in [10] which states that is $f(x)$ for a continuous function $a < x < b$ and assuming a piecewise continuous derivative on $[a, b]$, let $\{y_n(x)\}$orthogonal polynomials relative to classical weight $\omega(x)$, if the integrals:

$$\int_a^b f^2(x)\, \omega(x)\, dx \tag{7}$$

$$\int_a^b (f-(x))^2\,\sigma(x)\,\omega(x)\,dx$$

are uniformly convergent, then the function $f(x)$ has a development following polynomials $\{y_n(x)\}$ and the series:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x) \tag{8}$$

is uniformly convergent in $x$ on any segment $[x_1, x_2]$ included in $[a, b]$ with

$$c_n = \frac{\int_a^b f(x)\,y_n(x)\,\omega(x)\,dx}{\int_a^b y_n^2(x)\,\omega(x)\,dx} \tag{9}$$

We will propose later in this article, modeling by Chebyshev polynomials.

### 3. Chebyshev Polynomials

Chebyshev polynomials are orthogonal set of functions recursively defined on the interval [–1, 1]. In two kinds. The Chebyshev polynomials of the first kind are defined by the recurrence relation $T_0(x) = 1$, $T_1(x) = x$, respectively

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \text{ for } n \geq 1 \tag{10}$$

Chebyshev polynomials of second kind are defined by $U_0(x) = 1$, $U_1(x) = 2x$, respectively

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \text{ for } n \geq 1 \tag{11}$$

In this paper, we will use the Chebyshev polynomials of first kind, whose interesting properties make them very attractive for the design of filters and for optimal polynomials interpolation. They form a complete orthogonal set in the interval [-1, 1] with respect to following the weighting function:

$$\omega(x) = \frac{1}{\sqrt{1-x^2}} \tag{12}$$

In figure 1 are plotted curves of some first kind Chebyshev polynomials.

The Chebyshev polynomials also satisfy a discrete orthogonal relation. If $x_k$ ($k$ =1, 2,...$m$) are the $m$ zeros of $T_m(x)$, and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k)\,T_j(x_k) = \begin{cases} 0 & \text{if } i \neq j \\ m/2 & \text{if } i = j \neq 0 \\ m & \text{if } i = j = 0 \end{cases} \tag{13}$$

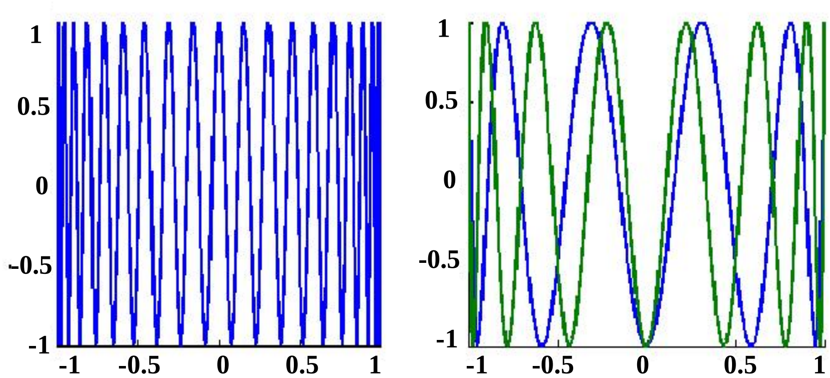


Figure 1. Curves of some Chebyshev polynomials of first kind

The trigonometric form of the Chebyshev polynomials of first kind is given by

$$T_n(x) = \cos(n \cos^{-1}(x)) \quad (14)$$

These polynomials are closely related to cosine trigonometric functions [11]. The zeros of $T_n(x)$ are derived from as (14) $T_n(x_j) = \cos(\arccos(x_j)) = 0$ as which implies the following:

$$x_j = \cos\left(\pi \frac{2j-1}{2n}\right), \; 1 \le j \le n \quad (15)$$

There are exactly $n$ distinct zeros of $T_n(x)$ in [–1, 1]. The Chebyshev polynomials also satisfy a discrete orthogonality relation. If $x_k =$ ($k$ = 1, 2,...$m$) are the $m$ zeros of $T_m(x)$, and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k)\, T_j(x_k) = \begin{cases} 0 & \text{if } i \ne j \\ m/2 & \text{if } i = j \ne 0 \\ m & \text{if } i = j = 0 \end{cases} \quad (16)$$

The extreme of $T_n(x)$ are also derived from equation (14) as $T_n(y_j) = \cos(\arccos(y_j) = \pm 1$, thus

$$y_j = \cos(\pi \frac{j}{n}), \quad 0 \le j \le n \quad (17)$$

At all of the maxima, $T_n(x) = 1$ while at all of the minima, $T_n(x) = -1$. This is the property that makes the Chebyshev polynomials extremely useful in polynomial approximation of functions. Many other properties of Chebyshev polynomials can be found in [12].

Let us expand a signal $s(t)$ in terms of Chebyshev polynomials series, that is,

$$s(t) = \sum_{k=0}^{n} c_k T_k(t) \quad (18)$$

The coefficients $c_k$ are calculated as follow:

$$c_k = \frac{\langle s, T_k \rangle}{\langle T_k, T_k \rangle} = \frac{\int_{-1}^{1} \frac{s(t)\, T_k(t)}{\sqrt{1-t^2}} dt}{\int_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}} dt} \quad (19)$$

$$= \frac{1}{d_k^2} \int_{-1}^{1} \frac{s(t)\, T_k(t)}{\sqrt{1-t^2}} dt$$

where $d_n^2 = \begin{cases} \pi & \text{if } i = j \\ \pi/2 & \text{if } i \ge j \end{cases}$

Gauss-Laboto method is a powerful tool for numerical integration, especially dedicated to orthogonal polynomials [12]. Gauss quadratures method for numerical integrations easy the evaluation of coefficients $c_k$. It stipulates that for a given family of orthogonal polynomials ($y_n(x)$) in a real interval [$a$, $b$], with respect to weight function $\omega(x)$, the following approximation holds:

$$\int_a^b f(x)\, \omega(x)\, dx \approx \sum_{j=1}^{M} G_i\, f(x_j) \quad (20)$$

Where $f(x) \in L^2[a, b]$, $x_j$ are roots of $y_m(x)$ and $G_j$ are called Christoffel numbers. Equation (20) is Gauss quadratures formulae; it is exact for all polynomials of degree inferior or equals to $2M$–1. Applying Gauss-Lobatto integration method on Chebyshev polynomials let to

$$\int_{-1}^{1} \frac{z(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n} \sum_{j=1}^{n} z(x_j) \quad (21)$$

Where $x_j$ are roots $T_n(t)$ given by (15) and all the Christoffel numbers are equal to $\pi / 2$. To compute $c_k$ in (1913), we use zeros of $T_{n+1}$.

**4. Modeling Approach**

The block diagram in Figure 2 shows the general principle of decomposition and synthesis of the ECG with the orthogonal polynomials. In the decomposition phase, the ECG signal is first divided into portions (windows) that we call blocks, in addition, a signal that is decomposable within Chebyshev base polynomials must be a function of $L^2[-1,1]$ to satisfy the above condition. Each block $s(t)$, $t \in [0, t_B]$ is of finite energy and should be transposed into $[-1,1]$ domain by a simple linear transformation as follows:

$$x = -1 + \frac{2}{t_B} t \qquad (22)$$

Where $t_B$ is the duration of the sampled (into blocks) of signals.

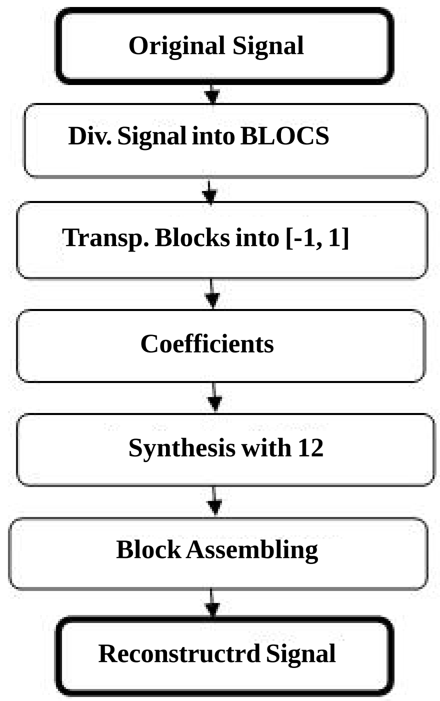


Figure 2. The signal processing chain for a complete transmission session

The polynomial transformation consists in determining the polynomial coefficients for each signal segment. All methods of polynomial decomposition of the ECG signals proposed so far segment the signal into blocks that coincide exactly with the cardiac cycle [13]. In such schemes, a preliminary step, that consists in the detection of QRS complexes is necessary to achieve correct segmentation. For Discrete Chebyshev Transform instead, it is possible to use blocks signals made of multiple cardiac cycles.

There is no requirement on the positions of the ECG's characteristic waves inside a block. Thus, the segmentation can be carried out blindly; only the duration of blocks must be specified. The next stage is the modeling mechanism consisting in the certain steps: segmentation, decomposition into the basis of Chebyshev polynomials and the calculation of coefficients.

The signal reconstruction stage consists in two steps the synthesis of the signal and the blocks assembling.

**5. Results and Discussions**

As the validity of the method is depending on the efficiency of the signal approximation with the polynomial we provide in the following some evaluations for Chebyshev polynomials and the Discrete Chebyshev Transform.

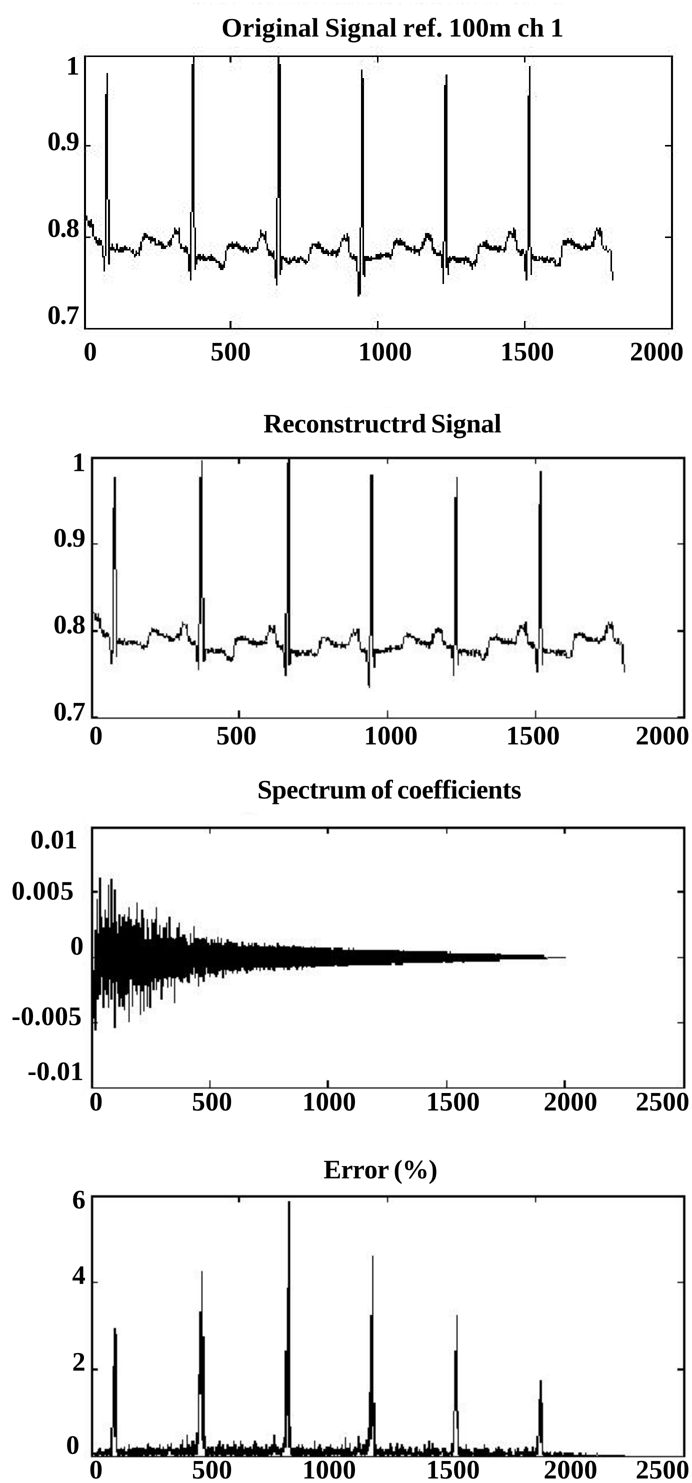


Figure 3. Applying DChT algorithm to record ref. 100, channel 1 using Chebyshev polynomials up to degree 2000.

We conducted our numericexperiments in Matlab environment, using signals from the MIT - BIH arrhythmia database [14], and also records available online [15]. Each record consists of two channels of signals. These signals are sampled at a rate of 360 Hz and use 11 bits/sample resolution. The modeling efficiency is measured using the mean square error (MSE), which is expressed as follows:

$$MSE = \frac{1}{n}\sum (\hat{S}_n - S_n)^2 \times 100 \tag{23}$$

For instance, we show on figure 3, the original and reconstructed of 5 seconds (i.e. 1800 samples) signals of record number 100, channel 1 from the MIT-BIH data base. On the top is the original signal, in the middle is the reconstructed signal and the spectrum of polynomial coefficients is plotted at the bottom. Chebyshev polynomials up to degree 2000 were used for that matter.

Both the original signal shown in figure 4 correspond to medically abnormal ECG. We can appreciate the strength of Discrete Chebyshev Transform as to faithfully reproduce the abnormalities included in the ECG signal. We applied the Discrete Chebyshev Transform over 40 signals from the MIT database [14] [15]. The method is proven sufficiently robust in all circumstances.

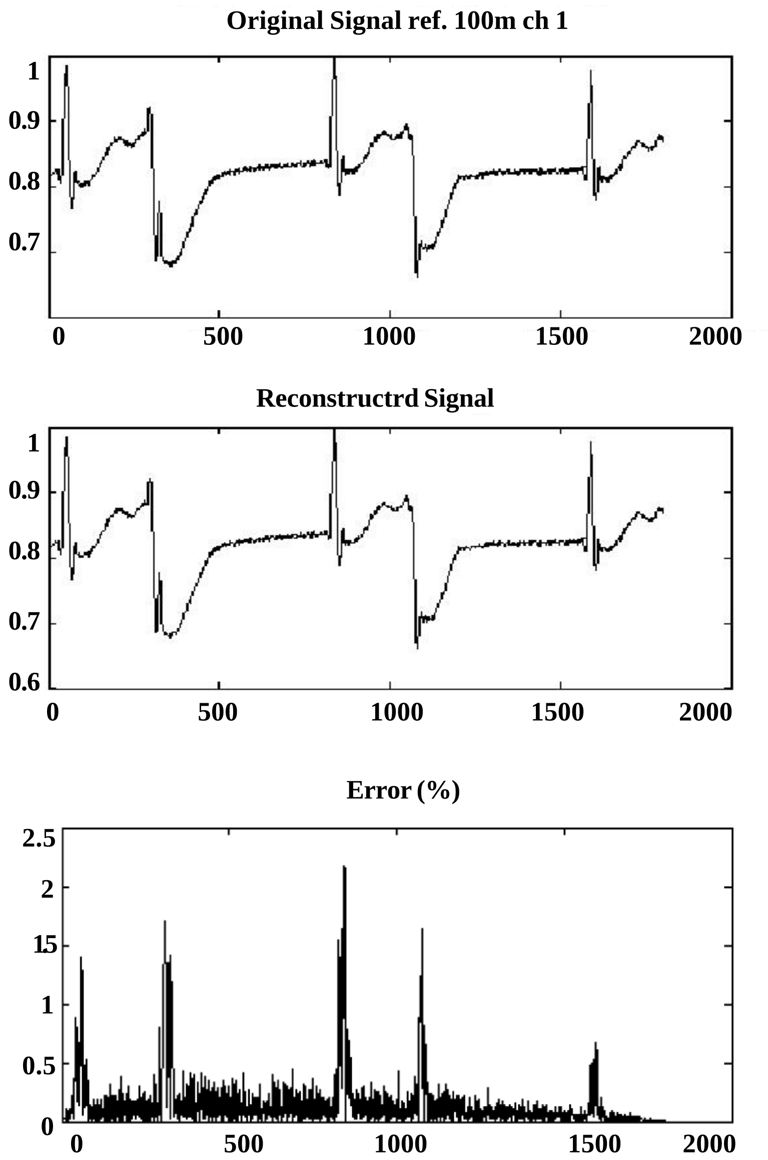


Figure 4. Examples of Discrete Chebyshev Transform compression of ECG signals (Signals ref. 207)

It is shown in Figure 5, the variation curves of the MSE as a function of the order *n* of Discrete Chebyshev Transform. It should be noted that the signals with high values of n are those incorporating very sharp impulses in their QRS complexes. As already highlighted in Figure 5, these regions of the QRS complexes contribute much more than other parts of the signal in the formation of the reconstruction errors. For which the values of the*n*are very large, the reconstructed signals remain faithful to the originals, the changes occur only at the amplitudes of QRS complexes.

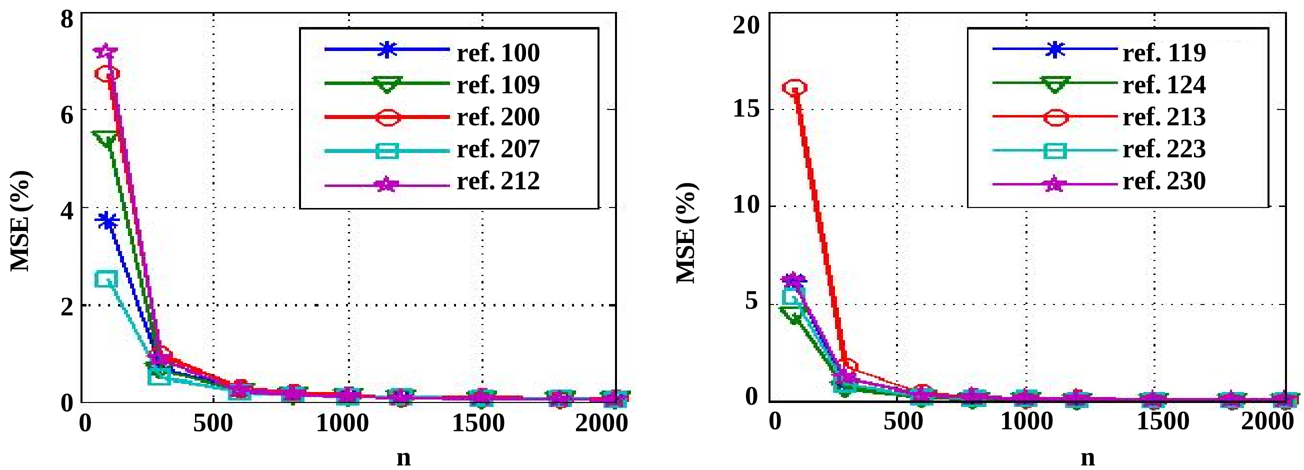


Figure 5. Variation curves of the MSE as a function of the order *n* of Discrete Chebyshev Transform for different reference signals

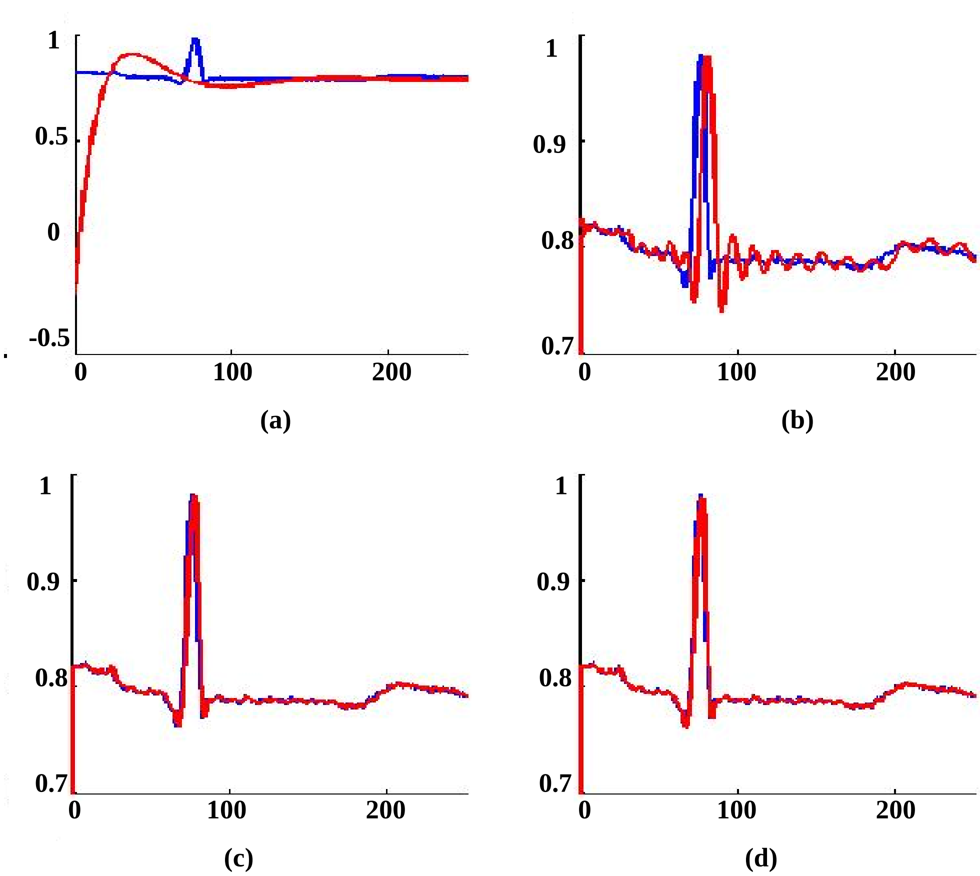


Figure. 6. Zoom on the original and reconstructed signals of signal referenced 100 for different values of order *n*: a)*n* = 20, b)*n* = 200 c) *n* =1000, d) *n* = 2000.

Zooming the graphics in the case of signal referenced 100. It can be seen in figure 6(c) that the coincidence of the reconstructed signal with the original signal is acceptable for *n* very great, It can be seen in Figure 6(d) that the coincidence of the reconstructed signal with the original signal is almost perfect (*n* = 2000).

## 6. Conclusions

Mathematical methods of polynomial interpolation and polynomial approximations inspired us to develop a modeling algorithm for ECG signals. This is based on the principle of signal expansion in series of Chebyshev polynomials.

The principle of ECG data modeling is presented and the relevant examples for the capability of the method are provided. The first results obtained exhibit the faithfully reproduction abnormalities included in the ECG signal. As future works, we would like making the implementation of algorithm in micro-system.

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## Authors Biography

***Fatima Guendouzi*** received her Master in automatic and signal processing degree from Jijel University Algeria in 2010. She is currently a PhD student at Laboratory of Instrumentation (LINS), USTHB, Algeria. Her research interests are biomedical signal and signal processing.

***Mokhtar Attari*** received the Engineer degree in Electronic Engineering from the Ecole Nationale Polytechnique of Algiers in 1985. He received his Master degree (Highest ranking) in 1987 and the Ph.D degree in 1991 in instrumentation and measurement from Joseph Fourier University, INP-Grenoble, France. Since 2002 he becomes Professor of Electrical Engineering at the faculty of Electronics and Computers. He is the co-founder and the actual Director of the laboratory of instrumentation. His present research interests focus on Sensors and analog conditioning design, Neuronal technique for sensor correction, A/D converters modeling and architecture, biomedical instrumentation including physiological signals and rehabilitation medicine.